XI. *On Autopolar Polyedra.*

By the Rev. Thomas P. Kirkman, *A.M.*, *F.R.S.*, *Rector of Croft with Southworth.*
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I.* As any face of a p-edron N is named from the number m of its edges an m-*gon*, so from the number n of its edges I call a summit of N an n-*ace*; and a polyedron having k faces and l summits may be denominated a k-*edron* or an l-*acron*, as may be most convenient.

By an *autopolar polyedron*, I mean one in which the number, rank, and collocation with respect to an α-gon A of its summits and remaining faces, are exactly the number,

* The following note, "On the Analytic Problem of the Polyedra," is kindly placed at my disposal by Arthur Cayley, Esq., who has wisely judged that to this investigation of a subject so new and intricate, some such statement should appear by way of introduction. I doubt not that the reader will approve of my appending it as he has written it. The note comprises a clearer statement of some things which may be found in my memoir "On the Representation and Enumeration of Polyedra," in the twelfth volume of the Memoirs of the Literary and Philosophical Society of Manchester, with some matter which has arisen in our correspondence; and particularly it supplies a defect in my statement of analytic conditions in art. 22 of that memoir, which Mr. Cayley with his rare penetration was the first to point out and amend. I have there laid down, that "multiplets are to be made with a symbol, under these two conditions: first, that every contiguous pair of symbols in any multiplet shall be a contiguous pair in some one other; and secondly, that no three symbols in any multiplet shall occur in any other." It should have been laid down, as Mr. Cayley here states it, that no contiguous duad shall occur non-contiguously, and that no non-contiguous duad shall be twice employed. By the words, *in some one other*, any reader of my memoir will see that I meant, *in some one other only*.

"*The Problem of Polyedra.*

"Let a, b, c, d, e, f, g, h, &c. represent the vertices of a polyedron, then a face will be represented *e. g.* by $abcde$, where the contiguous duads, viz. ab, bc, cd, de, ea are the edges of the face; and calling the face K, we may write

$$K=abcde. \qquad (1.)$$

"It is to be noticed that the letters of a face-symbol may be taken forwards or backwards from any letter without altering the meaning of the symbol. Thus, $abcde$, $bcdea$, &c., $edcba$, &c. might any of them be taken to denote the face K. The diagonal of a face cannot be either an edge or a diagonal of any other face, *i. e.* a non-contiguous duad such as ac in face-symbol K cannot be a duad contiguous or non-contiguous of any other face-symbol. But each edge of a face must be an edge of one and only one other face, *i. e.* each contiguous duad such as ab in the face-symbol K must be a contiguous duad of one and only one other face-symbol L. And moreover two faces cannot have more than a single edge in common, *i. e.* two face-symbols cannot contain more than a single contiguous duad the same in each symbol.

"The face K contains the edges ab, ac, i. e. the edge ab is contained in the face K; it will also be contained in one and only one other face, suppose L; this face will contain another edge through the vertex a, suppose rank and collocation with respect to at least one α-ace a in it, of its faces and remaining summits. This summit a may be considered the pole of that face A. A p-edral p-acron, whose p summits are the poles of its p faces, is an autopolar polyedron.

To every edge AB, between the α-gon A and β-gon B, in such a figure, corresponds an edge ab, between the α-ace a and the β-ace b, and *vice versâ.* Two such edges I call a *gamic pair*, or a pair of gamics, either being the gamic of the other. If these edges

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the edge af, and so on, until we arrive at a face containing the edge ae; we have, for example,

$$
\begin{aligned}
K &= eabcd \\
L &= baf\,.. \\
M &= fag\,.. \\
N &= gah\,... \\
P &= hai\,.. \\
Q &= iae\,...
\end{aligned}
$$

and we thence derive the vertex-symbol

$$a = \text{KLMNPQ}, \qquad (2.)$$

where the contiguous duads KL, LM, MN, NP, PQ, QK represent in order the edges through the vertex a. The remarks before made with respect to the face-symbols apply to the vertex-symbols. A non-contiguous duad such as KN of the vertex-symbol a cannot be a duad contiguous, or non-contiguous of any other vertex-symbol, but each contiguous duad such as KL of the vertex-symbol a must be a contiguous duad of one and only one other vertex-symbol b. And the symbols of two vertices cannot contain more than one contiguous duad the same in each symbol.

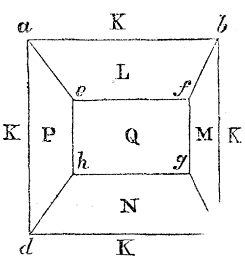
"Any edge of the polyedron admits of a double representation; it is the junction of two vertices, or the intersection of two faces. Thus ab and KL will represent the same edge, or we may write

$$ab = \text{KL}. \qquad (3.)$$

"It is to be remarked, that in this system each to each equation ab=KL; there corresponds one and only one equation of the form ae=KQ, *i. e.* to an edge considered as drawn from a given vertex in a given face there corresponds one and only one other edge from the same vertex in the same face.

"It has been shown how the system of face-symbols (1) leads to the system of vertex-symbols (2) and the system of edge symbols (3); and generally, any one of the three systems leads to the other two; and the three systems conjointly, or each system by itself, is a complete representation of the polyedron. As an example, take the hexaedron: the three systems are,—

K=$abcd$ (1)	a=LPK (2)
L =$abfe$	b=LMK
M=$bfge$	c=MNK
N =$gcdh$	d=NPK
P =$dhea$	e=LPQ
Q=$hefg$	f=LMQ
	g=MNQ
	h=NPQ



ab=KL	ae=PL	ef=LQ (3)
bc=KM	bf=LM	fg=MQ
cd=KN	cg=MN	gh=NQ
de=KP	dh=NP	he=PQ

"Consider, now, two polyedra having the same number of vertices and also the same number of faces.

ab and AB meet at a summit *b*, they are *nodal gamics* in the *nodal summit* b and the *nodal face* B. From the first notions of poles and polars, it is evident that no edge can be its own gamic.

Every polyedron is either *autopolar* or *heteropolar*. A heteropolar *p*-edral *q*-acron H is the *sympolar* of a *q*-edral *p*-acron H′, whose summits are the poles of the faces of H,

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And let the vertices and faces of the first polyedron taken in any order be represented by

$$abcde\,..\,\mathrm{KLM}\,..,$$

and the vertices and faces of the second polyedron taken in a certain order be represented by

$$a'b'c'd'e'\,..\,\mathrm{K'L'M'}\,..$$

Then, forming the substitution symbol

$$a'b'c'd'e'\,..\,\mathrm{K'L'M'}\,..\,abcde\,..\,\mathrm{KLM}\,..,$$

which denotes that a' is to be written for a, b' for b ... K′ for K, &c., if, operating with this upon the symbol system of the first polyedron, we obtain the symbol system of the second polyedron, the second polyedron will be syntypic with the first. It should be noticed, that there may be several modes of arrangement of the vertices and faces of the second polyedron, which will render it syntypic according to the foregoing definition with the first polyedron, *i. e.* the second polyedron may be syntypic in several different ways with the first polyedron. This is, in fact, the same as saying that a polyedron may be syntypic with itself in several different ways. Suppose, next, that the number of vertices of the second polyedron is equal to the number of faces of the first polyedron, and the number of faces of the second polyedron is equal to the number of vertices of the first polyedron. And let the vertices and faces of the first polyedron in any order be represented by

$$abcde\,..\,\mathrm{KLM}\,..,$$

and the faces and vertices of the second polyedron in a certain order be represented by

$$\mathrm{A'B'C'D'E'}\,..\,k'l'm'\,..$$

Then, forming the substitution symbol

$$\mathrm{A'B'C'D'E'}\,..\,k'l'm'\,..\,abcde\,..\,\mathrm{KLM}\,..,$$

if, operating with this upon the symbol system of the first polyedron, we obtain the symbol system of the second polyedron, the second polyedron is said to be polar-syntypic with the first; and, as in the case of syntypicism, this may happen in several different ways.

"Lastly, if there be a polyedron having the same number of vertices and faces, and if the vertices and faces in any order be represented by

$$abcd\,..\,\mathrm{KLMN}\,..,$$

and the faces and vertices in a certain order be represented by

$$\mathrm{ABCD}\,..\,klmn\,..$$

Then, forming the substitution symbol

$$\mathrm{ABCD}\,..\,klmn\,..\,abcd\,..\,\mathrm{KLMN}\,..,$$

if, operating with this upon the symbol system of the polyedron, we reproduce such symbol system, *i. e.* in fact, if the polyedron be polar-syntypic with itself, the polyedron is said to be *autopolar*; and in accordance with a preceding remark, this may happen in several different ways. It is clear that the substitution symbol, operating on the symbol system of the vertices, must give the symbol system of the faces, and conversely; but operating on the symbol system of the edges, it must reproduce such symbol system of the edges: and this last condition will by itself suffice to make the polyedron autopolar, *i. e.* the polyedron will be autopolar if the substitution symbol, operating on the symbol system of the edges, reproduces such symbol system."

and whose faces are polar to the summits of H. Every edge of H is the gamic of an edge of H′, and H and H′ are either the *sympolar* of the other, and form a *sympolar pair*.

II. An edge of a polyedron is said to *convanesce*, when its two summits run into one, and it is said to *evanesce*, when its two faces revolve into one.

An edge AB is said to be *convanescible*, when neither A nor B is a triangle, and AB joins two summits, which have not besides A and B two faces, one in each summit, collateral nor having a common summit.

An edge *ab* is said to be *evanescible*, when neither *a* nor *b* is a triace, and the two faces about *ab* are not, one in each, in two summits, besides *a* and *b*, collateral nor in one face.

If AB be convanescible, its gamic *ab* is evanescible, and *vice versâ*; and the pair AB, *ab* are *vanescible gamics*, when in an autopolar polyedron.

III. Theorem. *No polyedron, not a pyramid, has every edge both in a triangle and in a triace.*

For let it be supposed that P, which is not a pyramid, has every edge both in a triangle and in a triace, and let E be a face of P not less than the greatest; the faces collateral with E are by hypothesis all triangles. Let A and B be two of these having two contiguous edges of E. There will be a summit of E not a triace, otherwise P would be a pyramid; let this be *d*, a summit of E, A and V (EAV), V being a face collateral with A, but not with E. As one extremity of the edge AV is not in a triace, the other is; therefore AV will end at the triace AVW, the vertex of A. As P is no pyramid, it has more summits than one not in its base E, which are all connected by edges; therefore the edge AW must pass from AVW to some summit not in E.

The edge EA having an extremity *d* not in a triace, terminates at the triace EAB, where AB is the only edge meeting E, and the triangles A and B have a common vertex AVB, the intersection of AV and AB. But this is the triace AVW; therefore AW is AB, an edge passing through a summit of E, contrary to what has been proved. Q. E. A.

Therefore P has an edge either not in a triace, or not in a triangle. Q. E. D.

IV. Theorem. *Every polyedron* P, *not a pyramid, has either a convanescible or an evanescible edge.*

For P has either the edge AB, whose faces A and B are neither of them triangles, or the edge *cd*, whose summits are neither of them triaces. Let it have AB, an edge not in a triangle, lying between the summits *e* and *f*. By definition of a convanescible edge, if *e* and *f* have not besides A and B two collateral faces, C and D, AB is a convanescible edge of P. C and D may have a common edge *gh*, of any length, or a point.

If *e* and *f* have two collateral faces, C through *e*, and D through *f*, intersecting in *gh*, then neither *e* nor *f* will be a triace, or else one of them, *e*, will be a triace. If, firstly, neither be a triace, AB=*ef* is not in a triace, and is therefore evanescible, unless A and B about it are in two other summits *k* and *l* collateral or in one face. Now if a section be made through *ef*, and *m*, any point of *gh*, it will cut any edge *kl* drawn or drawable upon P, from *k* in A to *l* in B, and will divide P into P′, containing A, and P″, containing B. But this section is the triangle *efm*, whose sides are *em* in C and *fm* in D; therefore *m*, any point of *gh*, is a point of *kl*; which is absurd, unless *kl* is *gh*. And if *kl* is *gh*, *m* may be a point indefinitely near to *k*, and the section *efm* will be the face A, which is impossible, because A is no triangle. Wherefore the absurdity remains, and *kl* has no existence. Therefore P has the evanescible edge AB.

And if, secondly, *e* be a triace, AC and BC are edges of C, *ep* and *ep'*, and C having three summits in A and B, *epp'*, and one or two in *gh*, is no triangle; therefore *ep'* is an edge of P not in a triangle, which is convanescible, unless *e* and *p'* are in two covertical faces besides C and B. Now if they are, A is one of those faces, *e* being the triace ACB. Let G be a face through *p'* meeting A on P. The triangular section *efm* being made as above, cuts no face besides C and D; and divides P into P' containing A, and P'' containing B with its edge *ep'*. Therefore G, being uncut and covertical with A, is entirely in P'; and being uncut and having the summit *p'*, is entirely in P''. Which is absurd; wherefore G has no existence, and P has the convanescible edge *ep'*.

Thus it is proved that in any case P, being not a pyramid, and having an edge AB, not in a triangle, has either a convanescible or an evanescible edge.

Next, let P have an edge *cd*, not in a triace. Then will $P_{,}$, the sympolar of P, have an edge CD, the gamic of *cd*, not in a triangle, and also either a convanescible or an evanescible edge ε; wherefore P has the gamic of ε, which is either evanescible or convanescible.

Therefore in all cases P, not being a pyramid, has a vanescible edge. Q. E. D.

V. Theorem. *Any* p-*edral* q-*acron* P, *not a pyramid, can be reduced by the vanishing of an edge to either a* (p−1)-*edral* q-*acron or a* p-*edral* (q−1)-*acron.*

For P must have either a convanescible edge AB, or an evanescible edge *ab*. Let it have the former (AB).

If AB convanesces by the union of its summits, P becomes P', losing the faces A and B, and receiving two others, A' and B', in their place, A' being A with one edge less, and B' being B with one edge less. At the same time the two summits *dd'* of (AB) disappear from P, which in their stead and at their union, receives a new summit *h* having two edges less than the sum of theirs. All the faces of P containing *d* or *d'* remain unchanged, except that each now contains the summit *h*, whose edges are those of *d* and *d'* together save two; and no two of these faces thus brought to have a common summit *h* can coincide, because, by the definition of the convanescible *e*, they had not in P a common edge. Hence all the other edges of P remain undisturbed in the result P'. Or if *ab* be an evanescible edge of P, and evanesces by the revolution of DD' its two faces into one plane, giving rise to a new face H, this will have the edges of D and D' together save two; the summit *a* becomes *a'* with one edge less, and *b* becomes *b'* with one edge less. All the summits of P containing D and D' remain unchanged, except that each now contains the face H; and no two of these summits, thus brought into a common face H, coincide, because by the definition of (*ab*) they were not before in any line drawn or drawable on P. Hence all the other edges of P remain unaltered.

It is evident that P by the vanishing of AB, becomes P′ a p-edral $(q-1)$-acron; or by the vanishing of ab becomes P′ a $(p-1)$-edral q-cron. Q. E. D.

If P′ be not a pyramid, it can further be reduced by the evanescence or convanescence of an edge, until the result of the reduction is a pyramid Π. If there be *no pyramid of higher rank than* Π, to which P can be thus reduced, P will be said to be *generable from* Π, and can evidently be *generable* from none but Π.

When P is autopolar, it has both a convanescible and its evanescible gamic; and thus, from being a p-edral p-acron, it reduces to P′, a $(p-1)$-edral $(p-1)$-acron, which will also be autopolar; and P is *generable* only from one pyramid Π.

VI. The problem of the enumeration of N-edra is thus reduced to this: *To find how many* N-*edra are generable from the* K-*edral pyramid* Π. The solution of this question is to be founded on the consideration that if P is generable from Π, it can be constructed by the introduction into Π of new convanescible and evanescible edges; whether P be autopolar or heteropolar. I shall first consider the *autopolars*, and afterwards the *heteropolars* generable from the pyramid.

Problem. *To find the number of nodally autopolar* (r+2)-*edra generable from the* (r+1)-*edral pyramid.* (See below the definition of nodally autopolar.)

Let the r triangles and r triaces of the pyramid Π be numbered thus,

$$\begin{array}{ccccccccc} \mathbf{1}\ \mathbf{2} & & \mathbf{3} & & \ldots(\mathrm{R}-3) & & (\mathrm{R}-2) & & (\mathrm{R}-1)\ \ \mathrm{R}\ , \\ & r\ (r-1) & & (r-2)\ldots & & 4 & & 3 & \ \ 2\ \ 1 \end{array}$$

the upper line denoting the faces as they are read by an eye within the pyramid, and the lower the summits.

If E_r and E_l signify the two summits on the right and left of the eth triangle, and e_r e_l be the faces on either side of E,

$$\mathrm{E}_l=2-e\pm$$
$$\mathrm{E}_r=1-e\pm,$$

where e and E are the same number, as are r and R.

The $\pm$ in this and all functions of these signatures, denotes that such a multiple of r is to be added or subtracted as shall cause any signature to have an integer value between 1 and r inclusive.

This arrangement is autopolar; for the vertex ω is the pole of the base Ω, and the nth triangle is placed with respect to the adjoining faces and summits and their signatures, as the nth summit with respect to the adjoining summits and faces.

If two gamic edges meet in the kth face at the kth summit, we must have either

$$k=2-k\pm \text{ or } k=1-k\pm;$$

i. e. $$k=1, \text{ or } k=\tfrac{1}{2}(r+2), \text{ or } k=\tfrac{1}{2}(r+1).$$

This shows that there are only two *nodal summits*, which have the signatures 1 and $\frac{1}{2}(r+2)$ when r is even, and 1 and $\frac{1}{2}(r+1)$ when r is odd.

It will be convenient to call the line joining the nodal summits, *nodal diagonal* or *nodal line.* When r is even, the nodal diagonal is a diameter; but not when r is odd. When r is odd, the line which bisects the nodal diagonal *at right angles* will be known to us as the *nodal axis*, or the *axis of symmetry*. It passes through one summit of the base, and bisects the side opposite to that summit.

When r is even, there is no method of autopolar signature of the pyramid, which shall not exhibit a nodal diagonal. This pyramid is only *nodally autopolar*. But when r is odd, the signatures may be *enodal*, if every summit e is made the pole of the face E opposite to it. This odd-angled pyramid is either *nodally* or *enodally autopolar*. The autopolarity of the even-angled pyramid is *purely nodal*.

If, now, any two non-contiguous summits numbered e and f of this base be joined by the diagonal ef, and the two portions O and O$'$ of Ω be considered as faces about ef, the summits e and f are tessaraces in the edge OO$'$. The vertex ω must, therefore, if autopolarity is to be kept, become two summits o and o', and the triangles numbered EF will be quadrilaterals intersecting in the edge oo'. E and F, be it remembered, are the two numbers e and f.

The two edges, thus added to Π, are evidently a gamic pair, of which OO$'$ is evanescible, and oo' convanescible, and the result P is an autopolar $(r+2)$-edron.

VII. The question to be answered is, how many such results P can be made by drawing a *generator* OO$'$ between two summits of Ω, such that no two P shall be either identical, or one the reflected image of the other?

If ef and $e'f'$ be two generators OO$'$ of the same P, it is evident that $f-e=\pm(f'-e')$, since Ω is divided alike by both ef and $e'f'$. And the figure made by the points of the base $E_rE_lefF_rF_l$ will be exactly like that presented by the points $E'_rE'_le'f'F'_rF'_l$, or its reflected image.

For the only difference possible between the two results P, P$'$, generated by ef and $e'f'$, will consist in the mutual arrangement of the introduced 4-gons and 4-aces; and any difference herein will prevent P from being the same with P$'$.

The four following equations, along with

$$f-e=\pm(f-e'),$$

comprise all the possible conditions of similarity between P and P$'$:—

$$E_l-e=E'_l-e'\pm$$
$$E_l-e=e'-E'_r\pm$$
$$E_l-e=E'_l-f'\pm$$
$$E_l-e=e'-F'_r\pm.$$

The first affirms that the distance from the left-hand summit of E to e is the same as from the left of E$'$ to e', and measured in the same direction.

The second affirms the said distance is that from the right summit of E$'$ to e', measured in the opposite direction. In the first case looking at the configuration in Ω, the system $(e'f')$ is a direct repetition of the system (ef). In the other it is its reflected image.

The configuration of the third condition is that of the first, and of the fourth that of the second, with the difference that the signatures e' and f' have changed places in $e'f'$ in the third, and E′ and F′ have changed places about E′F′ in the fourth.

VIII. First, let, along with $e-f=\pm(e'-f')$,

$$E_l-e=E'_l-e'\pm,\ i.\ e.$$

$$2-2e=2-2e'\pm.$$

This, since $e=e'$ and $f=f'$ is contrary to hypothesis (for we are seeking a diagonal $e'f$ different from ef), means

$$2-2e=2-2e'\pm r,$$

or

$$e'=e\pm\tfrac{1}{2}r;$$

that is, r is even, and e' is the opposite extremity to e of the diameter of Ω through e. Hence f' is opposite f in the diameter through f.

It thus appears that when r is even, for every diagonal ef (not a diameter), there can be drawn a diagonal $e'f'$ in Ω, that generates the same $(r+2)$-edron P which is generated by ef; and that e' and f' are diametrically opposite to e and f. When ef is a diameter, it does not hence appear whether there is any $e'f'$ *a fellow-generator* with ef.

IX. Secondly, let

$$E_l-e=e'-E'_r\pm,\ i.\ e.$$

$$2-2e=e'-(1-e')\pm$$

$$2e+2e'=3+r\pm.$$

We gather that r is odd; for no multiple of r even can here be added to make this equation true. Hence

$$e'=\tfrac{1}{2}(r+3-2e+)$$

and

$$f'=\tfrac{1}{2}(r+3-2f+).$$

The diagonal $e'f'$ thus found is different from ef, unless $e'=f$, and $f'=e$, *i. e.* unless

$$2f+2e=3+r+,$$

the additional $+$ denoting either zero or $2r$. When this relation between f and e holds, it does not thus far appear whether a pair of generators, $e'f'$ and its gamic, can be drawn to produce the same P with ef and its gamic. This relation is otherwise thus:

$$e-1=-\left(f-\tfrac{1}{2}(r+1)\right),$$

or else

$$e-1=-\left(f-\tfrac{1}{2}(3r+1)\right),$$

showing that f is at the same distance measured backwards from the nodal summit $\tfrac{1}{2}(r+1)$ which is also $\tfrac{1}{2}(3r+1)=r+(\tfrac{1}{2}r+1)$, that e is measured forwards from the nodal summit 1. The line ef is either the *nodal diagonal* through 1 and $\tfrac{1}{2}(r+1)$, or it is parallel thereto. We may call all these diagonals *the nodal parallels.* It is evident that when ef and $e'f'$ are fellow-generators, ee' and ff' are nodal parallels; since

$$2e+2e'=2f+2f'=r+3+.$$

X. Thirdly, let

$$E_l - e = E'_l - f' \pm,$$

or

$$2 - 2e = 2 - e' - f' \pm,$$

which gives

$$2e = e' + f' \pm$$
$$2f = f' + e' \pm.$$

The only interpretation of this, consistent with $e'f'$ not ef, is

$$2e = e' + f'$$
$$2f = e' + f' \pm,$$

which can be true only when

$$2e - 2f = \pm kr;$$

but

$$2e \not> 2r, \text{ hence } k < 2,$$

and

$$2e - 2f = \pm r,$$

or

$$e = f \pm \tfrac{1}{2}r,$$

is the only possible relation; *i. e.* ef is the diameter of an even-angled base.

The equation $2e = e' + f'$ is $2e = 2e' \pm \frac{1}{2}r$, giving

$$e' = e \pm \tfrac{1}{4}r$$
$$f' = e' \pm \tfrac{1}{2}r = e \pm \tfrac{3}{4}r.$$

Therefore $r = 4m$.

By this we see that, when $r = 4m$, the diameters all pair themselves m summits apart into generators ef and $e'f'$, of the sum $(r+2)$-edron; for no value of e' and f' can coincide with e and f from which it is obtained, since

$$e' = f \text{ or } e' = e,$$

along with

$$e' = e \pm \tfrac{1}{4}r$$

and

$$f' = e \pm \tfrac{1}{2}r$$

cannot coexist.

XI. Lastly, let

$$E_l - e = e' - F'_r \pm,$$

or

$$2 - 2e = e' + f' - 1 \pm;$$

whence comes

$$2e + e' + f' = 3 +$$

and

$$2f + f' + e' = 3 + .$$

Of these, the only meaning that consists with $e'f'$, different from ef, is

$$2e + e' + f' = 3 + kr$$
$$2f + f' + e' = 3 + k'r,$$
$$2(e - f) = (k - k')r.$$

Now as neither e nor $f>r$, $(k-k')=\pm 1$, and

$$e=f\pm\tfrac{1}{2}r,$$

showing that ef is a diameter of an even-angled base, as is also $e'f'$, one case of ef. Putting $e'=f'\pm\frac{1}{2}r$, we deduce, from $k=1$, $k'=2$,

$$2e+2f'-\tfrac{1}{2}r=3+r$$

$$2e+2e'-\tfrac{1}{2}r=3+2r$$

$$2e+2f'=3+\tfrac{3}{2}r;$$

proving that $r=4h+2$, and

$$2e+2e'=3+\tfrac{5}{2}r.$$

These equations give e' and f' different from e and f, except when $e'=f=e\pm\frac{1}{2}r$, or $e'=e$; *i. e.* when

$$4e=\frac{6+3r}{2},$$

and

$$e=\tfrac{3}{8}(2+r);$$

or

$$4e=\tfrac{1}{2}(6+5r),$$

and

$$e=\tfrac{1}{8}(6+5r).$$

Hereby we learn that when $r=8h-2$, or $r=8h+2$, all the diameters of Ω pair themselves into generators, ef and $e'f'$, of the same $(r+2)$-edron P, except only that drawn in the first case through $e=\frac{3}{8}(2+r)$, and that drawn in the second through $e=\frac{1}{8}(6+5r)$.

XII. We can now easily enumerate the autopolar $(r+2)$-edra generable from the pyramidal $(r+1)$-edron (Π).

When $r=4h$, all the diagonals of Ω pair themselves into fellow-generators ef and $e'f'$. The number of diagonals is $\frac{1}{2}r.(r-1)-r=\frac{1}{2}(r^2-3r)$. Hence the number of autopolars (P) required is $\frac{1}{4}(r^2-3r)$.

When $r=4h+2$, all the diagonals but one, namely one of the diameters (XI.), pair themselves; consequently, the number of autopolars (P) sought is

$$\tfrac{1}{2}\{\tfrac{1}{2}(r^2-3r)+1\}=\tfrac{1}{4}(r^2-3r+2).$$

The question left unsettled in (IX.), as to whether the nodal parallels pair themselves into fellow-generators, is decided in the negative by silence of our formulæ in X. and XI. on the subject of $r=2h+1$. The diagonals of Ω for r odd pair themselves, unless when

$$2f+2e=3+r,$$

or

$$2f+2e=3+3r.$$

For every value of e, whether $r=4k+1$, or $4k-1$, these equations give f on a nodal and unpaired parallel, except when they give $e=f$, or $e=f+1$; *i. e.* unless

$$4e=3+r,$$

$$4e=3+3r,$$

which give $e=f$; and unless

$$4e=1+r,$$

or

$$4e=1+3r,$$

which give

$$f=e+1.$$

When $e=f$, the nodal parallel is simply the point e, an evanescent parallel. When $e=f+1$, it is an edge of Ω. From every other point e of Ω a nodal parallel can be drawn, which has no fellow-generator, and the number of these is $\frac{1}{2}(r-3)$. Therefore the number of diagonals of Ω when r is odd, all different generators, is

$$\tfrac{1}{2}\{\tfrac{1}{2}(r^2-3r)+\tfrac{1}{2}(r-3)\}=\tfrac{1}{4}(r^2-2r-3),$$

the number of autopolars (P), when r is odd, constructed.

XIII. But it is next to be determined, how many times P, thus generated by a diagonal ef, is identical with P′, generated by a diagonal hk different from ef*.

As no two operations on the base Ω have been alike, it is certain that P′ cannot be reduced to a pyramid on the exact base Ω by the operation whereby P is so reduced; but it is possible that P′ may be reducible to an r-gonal based pyramid, by the union of some two faces which are not O and O′. If so, P′ will have a convanescible edge different from oo'. This can be none else than OH or OK, or else O′H or O′K, as H and K are the only faces not triangles distinct from O and O′, which have not a common convanescible, but an evanescible edge.

If OH be convanescible, O is not a triangle; therefore $r>4$, and the two summits of OH, when united, must give a summit at least a pentace. Greater than a pentace it cannot be, since there can be no summit of P, neither o nor o', ampler than a tessarace, and this must adjoin a triace, because no diagonal hk in Ω can make two collateral tessaraces. The edge OH must then both be convanescible and have a tessarace, and be an edge of the base of a 6-edral pyramid. It is easily seen, either by trial or by inspection of

$$\mathbf{1}_5\mathbf{2}_4\mathbf{3}_3\mathbf{4}_2\mathbf{5}_1,$$

that the only diagonals fulfilling these conditions are 13 and 24 (or its fellow 25). The first gives a 7-edron P with three convanescible edges, oo', 15, 34, which reduces to a pyramid either on the base $12oo'5$, or on $234o'o$. The second gives a 7-edron P′ having two convanescible edges, oo' and 45, reducing to a pyramid on the base $123o'o$. And P, P′, are plainly not repetitions of each other. Therefore no deduction is to be made from the number of $(r+2)$-edra (P) constructed from the pyramid. If Π_1 be this number,

$$\Pi_1=\tfrac{1}{4}\{(r^2-3r)4_r+(r^2-3r+2)4_{r-2}+(r^2-2r-3).2_{r-1}\};$$

the number of $(r+2)$-edra (P) generable from the $(r+1)$-edral pyramid, where the circulator $4_r=1$ or 0, as r is or is not $4m$.

* It is clearly impossible that any of these P can be either a pyramid or reducible to a pyramid of higher rank than the $(r+1)$-edral Π.

In all these (P) there is but one *leading system*, i. e. system of vanescible gamic pairs whereby P reduces to a $(r+1)$-edral pyramid; except only the two just found, of which one has *two*, and the other *three*, *leading systems*.

These Π_1 $(r+2)$-edra (P), it is to be kept in mind, are all *nodally autopolar*. Whether there be any other *enodally autopolar* $(r+2)$-edra generable from the same pyramid, when r is odd, remains to be hereafter determined.

XIV. Problem. *To determine the number of nodally autopolar* (r+3)*-edra generable from the* (r+1)*-edral pyramid, by introduction of gamic pairs.*

For the solution of this problem it is necessary *to determine the number of pairs of diagonals not crossing each other that can be drawn in an* r*-gon.*

This number is less than that of pairs of diagonals; and of diagonals there are $\frac{1}{2}(r^2-3r)$. Therefore $\mathrm{R}_{r,2}$ the function of r required is of an order not higher than the fourth. It is evident that the function is of the form

$$\mathrm{R}_{r,2}=r.(r-3)(r-4)(ar+b),$$

and by trial,

$$\mathrm{R}_{5,2}=5.2.1.(5a+b)=5,$$

and

$$\mathrm{R}_{6,2}=6.3.2.(6a+b)=21,$$

viz. three pairs at every angle, and three parallel pairs; therefore

$$a=b=\tfrac{1}{12},$$

and

$$\mathrm{R}_{r,2}=\tfrac{1}{12}.r.(r-3).(r-4)(r+1);$$

which can be generalized into

$$\mathrm{R}_{r,k}=\frac{r^{k|1}}{\lfloor k+1}\cdot\frac{(r-3)^{k|-1}}{\lfloor k+2},$$

the number of ways in which k diagonals, none crossing each other, can be drawn in an r-gon.

Let ef and hk be one of these R_r pairs of diagonals of Ω which do not cross each other. If we draw them and consider them as new edges, Ω is divided into three faces OO′O″, and the summits $efhk$ become tessaraces if they are all different, or if $e=h$, e is a pentace and f and k are tessaraces. The summit ω of the pyramid, if the result is to be autopolar, is broken into three summits $oo'o''$, and the faces EFHK are either four 4-laterals, or a pentagon E and two 4-laterals F and K. The result Q is an autopolar $(r+3)$-edron.

XV. We have to determine how many different $(r+3)$-edra Q can thus be generated by a pair of diagonals ef and hk of Ω.

Of two diagonals not crossing each other in an even-angled base, it is plain that both cannot be diameters. Let ef and hk not cross, and let not these be fellow-generators; then if neither be a diameter, there is another pair $e'f'$ and $h'k$, the fellow-generators of the former two (VIII.), such that the points $e'f'h'k'$ are diametrically opposite to the points $efhk$. And if one of the former two ef be a diameter, and $h'k'$ be the fellow of

hk, the points $feh'k'$ are diametrically opposite to $efhk$. Consequently all distances measured on the summits of Ω between the signatures $efhk$, $E_rE_lF_r$, &c. will correspond exactly with distances between the signature $e'f'h'k'E'_r\ldots$ or else between the signatures $feh'k'F_r\ldots\ldots$; for if $\alpha-\beta$ be one of these distances, $(\alpha+\frac{1}{2}r)-(\beta+\frac{1}{2}r)$ is the corresponding distance. Consequently Q generated by ef and hk will not differ from Q′ generated by $e'f'$ and $h'k'$, when ef and hk are neither of them diameters, nor from Q′ generated by fe and $h'k'$, when ef is a diameter of Ω. Let, when r is even,

$$R_r=R'_r+R''_r,$$

R'_r being the number of pairs ef and hk which are not, and R''_r that of the pairs ef and $e'f'$ which are, fellow-generators. It is evident that with the pairs R'_r we cannot obtain more than $\frac{1}{2}R'_r$ autopolars Q. Nor fewer. For let $e''f''$ and $h''k''$ give the same Q with the pairs ef and hk, and $e'f'$ and $h'k'$. Q″ from the first has the same faces OO′O″ with Q from the second; and if hk and its gamic be erased in Q, and $h''k''$ and its gamic be erased in Q″, the results P and P″ will be identical; *i. e.* ef will be the fellow of $e''f''$. In like manner hk is proved to be the fellow of $h''k''$; but $e'f'$ and $h'k'$ are the fellows of ef and hk; wherefore $e'f'$ and $h'k'$ are not different from $e''f''$ and $h''k''$; contrary to hypothesis, which is absurd. Therefore $\frac{1}{2}R'_r$ is exactly the number of $(r+3)$-edra Q generable by the pairs R'_r.

Next, let ef and $e'f'$ be one of the pairs R''_r, the number of which is easily seen to be $\frac{1}{4}r.(r-4)$, since $r-4$ lines not diameters can be drawn from e.

There is no pair hk and $h'k'$ of these R''_r that can give the same Q with ef and $e'f'$, else ef would be proved the fellow either of hk or $h'k'$. Consequently the number of different Q obtainable from these R''_r pairs is R''_r. It follows that the number $\Pi_{\prime\prime}$ of $(r+3)$-edra Q generable when r is even from all the R_r pairs is

$$\Pi_{\prime\prime}=\tfrac{1}{2}(R'_r+2R''_r)=\tfrac{1}{2}\big(R_r+\tfrac{1}{4}(r^2-4r)\big)=\tfrac{1}{24}\big(r^2.(r-4)(r-2)\big).$$

But yet this number is subject to a doubt, which will be discussed presently.

XVI. When r is odd and >3, we have seen (IX.) that if ef and hk be any two diagonals whose fellow-generators are $e'f'$ and $h'k'$, ee', ff', hh', kk' are nodal parallels. If then ef and hk do not meet each other, neither will $e'f'$ and $h'k'$ meet each other. Both these pairs have therefore been counted in the R_r pairs of not-crossing diagonals. And these pairs give the same result Q; for if B$-$A be any distance measured on the summits of Ω in one between summits affected by the first pair, $\big(A-\frac{1}{2}(r+3)\big)-\big(B-\frac{1}{2}(r+3)\big)$ is the corresponding distance in the summits affected by the second pair.

Next, let ef and $e'f'$ be fellow-generators, which do not meet each other in the base Ω. They are opposite sides of a quadrilateral $ee'ff'$; and the resulting Q cannot be identical with that made by any other pair gh ij, for this would require that gh should be fellow to one and ij to the other of ef and $e'f'$.

The number of pairs ef $e'f'$ comprises, first, that of the quadrilaterals whose opposite sides are non-contiguous nodal parallels ee' and ff', of which nodal parallels there are $\frac{1}{2}(r-3)$ (XII.); and secondly, every pair of lines ef, ef' drawn from the point $(e=f)$, the evanescent parallel; and thirdly, every pair drawn from the extremities of that nodal parallel which is a side of Ω. That is, there are $\frac{1}{2}(r+1)$ of these parallels whose non-contiguous pairs are $\frac{1}{8}(r-3)(r-1)$, which is the number of pairs ef and $e'f'$, of fellow-generators which do not cross each other, and of the $(r+3)$-edra Q generated by such pairs.

We suppose next that ee' is one of the $\frac{1}{2}(r-3)$ nodal parallels, and that hk, having a fellow-generator $h'k'$, does not meet ee'. Then $h'k'$ will not meet ee'; for ee', parallel to two sides hh' and kk' of the 4-lateral hk, $h'k'$, does meet $h'k'$ unless it meets hk. And for a reason given in this article the $(r+3)$-edron generated by ee' and hk cannot differ from that obtained from $e'e$ and $h'k'$. Every Q obtained by a pair ee', hk, is obtained also from $e'e$, $h'k'$.

Now let ee' and ff' be two of the $\frac{1}{2}(r-3)$ nodal parallels. They do not meet; and they generate a $(r+3)$-edron Q which cannot be generated by any other pair ee', hh' or hh', ii'; for, if it could, this other pair would be two fellows of ee' and ff', which have no fellows. Thus each of the $\frac{1}{8}.(r-3)(r-5)$ pairs of nodal parallels will generate a distinct $(r+3)$-edron Q.

From all this it appears that the number $\Pi'_{\prime\prime}$ of different (Q) obtained from the R_r pairs of diagonals that do not cross each other, is ($r>3$ and $=2m+1$)

$$\Pi'_{\prime\prime}=\tfrac{1}{2}\{R_r+\tfrac{1}{8}.(r-3)(r-1)+\tfrac{1}{8}.(r-3)(r-5)\}$$
$$=\tfrac{1}{48}\{2r.(r-3)(r-4)(r+1)+3(r-3)(r-1)+3(r-3)(r-5)\}.$$

XVII. It appears, I say: and it is certain, that none of these Q obtained from operations in the base Ω, are alike in the relation between the faces OO′O″ and the other faces of the figure. Any two of them, Q′ and Q″, present different arrangements of the circuit of base summits $1\,2\ldots r$, which enclose those three faces; so that Q′ and Q″ cannot be reduced to a pyramid having the base Ω whose summits shall be those signed $1\,2\,..\,r$, by similar operations with vanishing gamic pairs. This is plain, because Q′ and Q″ have not been constructed by similar operations upon the base Ω and the vertex ω.

But it remains to be proved that Q″ has not three faces different from OO′O″, which by proper selection of vanescible gamic pairs may be brought to form an r-gonal base Ω''. If such a thing is possible, this Q″ may have the relation to the contour of that Ω'', which Q′ has to that of Ω; and thus the operations that gave us Q′ will have generated the same $(r+3)$-edron with those which gave us Q″, and *vice versâ*.

We are then to examine whether Q″ has three summits, not $oo'o''$, which can by convanescing edges be united to form an r-gonal summit ω''.

Now Q″ has no summits besides $oo'o''$, except $1\,2\ldots r$. Of the three summits of which we are in search, two, one, or none may be among $oo'o''$. The three may be $oo'e$, oee' or $ee'e''$.

There may be a convanescible edge oe, the edge F.(F+1). Of these two faces, F and F+1, neither can be a pentagon; for the lines fg, fh, evanescibles making the pentace f, can neither of them be $f(f+1)$, a side of Ω. If, then, oe convanesces, F and F+1 become triangles, and oo', oo'', which are their edges out of the base, are neither of them convanescible. Therefore $oo'e$ cannot be united.

There may be a convanescible edge ee', which is $e.(e+1)$, or FO′. If F be a pentagon, for a reason just given, it adjoins two triangles upon $(e-1)e$ and $(e+1)(e+2)$, so that oe nor oe', one of the sides of it, cannot be a convanescible edge. If F be a quadrilateral $oo'ee'$, its sides oe', $o'e$ will be in the triangle $oo'e$ when ee' convanesces. Therefore oee' cannot be united into one summit.

Our three sought summits will therefore be $(e-1)$, e, $(e+1)$. That $(e-1)e$ may convanesce, its faces F and O must not be triangles, neither can (F−1) and O′ about $e(e+1)$ be triangles. Therefore O and O′ together have at least six summits of Ω, which, supplying at least one summit more to O″, cannot be *simpler* than a heptagon. Neither can it be *ampler*; for e is at most a pentace, adjoining two triaces $(e-1)$, $(e+1)$, for a reason above given. And no diagonals have been drawn to make a third tessarace among $(e-1)$, e, $(e+1)$, supposing two of them tessaraces. That is, these three can unite at most to form a heptace ω''. We are then to examine whether two diagonals can be drawn in the heptagonal Ω to make two adjoining convanescible edges. The only way of doing this is by drawing either 62 and 63, or 72 and 53: in either case 65 and 67 are both convanescible, as well as $o'o''$ and $o'o$. The faces of the decaedron in the first result are

$$3456,\ 362,\ 2671,\ 1o''2,\ 2o''o'o3,\ 3o4,\ 4o5,\ 5oo'6,\ 6o'o''7,\ 7o''1;$$

and in the second,

$$345,\ 35672,\ 721,\ 7o''1,\ 1o''o'2,\ 2o'3,\ 3o'o4,\ 4o5,\ 5oo'6,\ 6o'o''7.$$

These are identical, if we exchange the signatures in the edges $5o$, $6o'$, and $7o''$.

XVIII. This decaedron is the only one of the autopolar $(r+1)$-edra Q above constructed which has two *leading systems*, i. e. two pairs of leading convanescible edges, and two pairs their gamics evanescible; and in this case alone have the operations (62 and 63) that gave us Q′ brought out the same polyedron with the operations (72 and 53) that gave us Q″. It is therefore necessary to deduct one from the (Q) enumerated, namely, one for the value $r=7$.

This number is consequently $(r>3)$, (XV. XVI.),

$$\Pi_{,,}+\Pi'_{,,}=\tfrac{1}{24}.r^2.(r-4)(r-2).2_r$$
$$+\tfrac{1}{48}\{2.r.(r-3)(r-4)(r+1)+3(r-3)(r-1)+3(r-3)(r-5)\}.2_{r-1}-0^{(r-7)^2}.$$

XIX. Our next step is to take up one of the $(r+2)$-edra P (XIII.), obtained by drawing a diagonal ef in the base Ω of the pyramid. P has two quadrilaterals E and F, in either of which F we can draw a diagonal, making a triangle F′, a side of which is oo', the gamic of OO′ in the base. This new diagonal in F is either $F_l o)$ or $(F_r o')$; and the triangle is either $F_r oo'$ or $F_l oo'$. Thus a new triangle is introduced between F and F+1, or else between F and F−1, about the line oo'. Consequently there must be a new triace between f and $(f+1)$, or between f and $(f-1)$ on the edge OO′. Thus the generator OO′ is drawn from a summit of Ω to the mid-point of a side of Ω, dividing the faces OO′ whose summits together make $r+3$.

We are to consider how many $(r+3)$-edra S can be obtained by such an operation in Ω.

The diagonal ef being drawn, becomes by the motion of the extremity f, $e(f\pm\frac{1}{2})$. If $e'f'$ be the fellow-generator of ef, $e'(f'+1)$ is that of $e(f+1)$ when r is even, and $e'(f'-1)$ that of $e(f+1)$ when r is odd. Two fellow-generators when drawn are symmetrical with respect to the nodal line, which is a diameter when r is even, but not when r is odd. Consequently, $e'(f'\pm\frac{1}{2})$ will be the fellow-generator of $e(f\pm\frac{1}{2})$ when r is even, and of $e(f\mp\frac{1}{2})$ when r is odd. Even when fe is the fellow of ef, or ef is an unpaired generator, $f(e\pm\frac{1}{2})$ is a different line from either $e(f\pm\frac{1}{2})$ or $e(f\mp\frac{1}{2})$. Consequently, every line λ that can be drawn in Ω from a summit to the mid-point of an edge, has a fellow-generator, except only when $e(f+1)$ is the fellow of ef, which happens when r is odd, and $e(f+\frac{1}{2})$ bisects Ω, being the nodal axis perpendicular to the nodal line. And there is plainly but one such fellow of λ, which being drawn shall be symmetrical with it as to the nodal line. Therefore every $(r+3)$-edron S, except one, that is generated by a line λ, will be generated also by a line λ'. Of lines λ there are $r.(r-2)$; hence the number of $(r+3)$-edra S is not greater than $\frac{1}{2}\{r.(r-2)+2_{r-1}\}$. Nor are there fewer that can be reduced to a pyramid on the exact base Ω. It is, however, to be inquired, whether some of these (S) may not have two faces not O and O′, which can by the vanishing of two edges be united into an r-gon Ω', giving a pyramid on the base Ω'.

But before we examine whether any of these $(r+3)$-edra S are repetitions of each other, it is desirable to ascertain how many of them are reducible to the $(r+2)$-edral pyramid. If any one of them, s, is so reducible by the vanishing of a single gamic pair, it will be a repetition of a $(r+3)$-edron P, generated by a diagonal in the $(r+1)$-gonal Ω; and will have a convanescible edge ε between summits whose united ranks x and y give $x+y=r+3$. That these two summits in s cannot be o and o' is certain; for the edge oo' is in a triangle. Then one only, or neither of these x-ace and y-ace, will be o or o'.

XX. First, let neither of them be o or o'; i. e. let ε be an edge of Ω. The greatest of x and $y \not> 5$; for $ef_{\prime}$ being drawn in s to the mid-point of a face H, E becomes a 4-lateral containing o and o', and η the pole of H becomes a 4-ace containing $f_{\prime}o$ and o', unless E=H, in which case it is a 5-gon, containing o, o', $f_{\prime}$ and two summits of Ω. As no summit or face, except e and E, is affected by $ef_{\prime}$ in this case, all the faces but E about Ω remain triangles, and therefore there is no convanescent edge ε. Wherefore $x \not> 4$, and y is either 4 or 3. First, let $x=4$ and $y=4$; whence $r=5$. We are to look for an edge convanescible and between two 4-aces in s made from the pentagonal Ω. The only 4-aces in s, neither o nor o', are e and η. Therefore $e\eta$ is the edge ε, and $e=\eta\pm1$. We see from

$$\mathbf{1}_5\mathbf{2}_4\mathbf{3}_3\mathbf{4}_2\mathbf{5}_1,$$

that in order to have e and η contiguous 4-aces, we must either draw the generator

$(1, 5-\frac{1}{2})$ or $(2, 5+\frac{1}{2})$. In the first case, 1 is a 4-ace and 2 a 4-lateral; in the second, 2 is a 4-ace and 1 a 4-lateral. In either case 1 and 2 are 4-aces, but in both (12) is an edge of the triangle 5, and therefore not convanescible. Therefore $\mathfrak{s}$ has no existence when $r=5$. Let, next, $x=4$ and $y=3$; then $r=4$.

We are to look in *s* made from the 5-edral pyramid for a convanescent $\mathfrak{s}$ not passing through *o* or *o'*. The simplest way of doing this is to write out the four results of drawing the four generators λ in the 4-gonal Ω. The eight edges of the pyramid are represented thus:

$$\begin{array}{llll} 4_0\mathbf{3}_1 3_4\mathbf{3}_2 & 4_0\mathbf{3}_2 3_3\mathbf{3}_3 & 4_0\mathbf{3}_3 3_2\mathbf{3}_4 & 4_0\mathbf{3}_4 3_1\mathbf{3}_1 \\ 3_1\mathbf{3}_4 3_2\mathbf{4}_0 & 3_2\mathbf{3}_3 3_3\mathbf{4}_3 & 3_3\mathbf{3}_2 3_4\mathbf{4}_0 & 3_4\mathbf{3}_1 3_1\mathbf{4}_0, \end{array}$$

where the first and third places in the quadruplet show the rank and signature of the left and right summits, and the second and fourth, those of the upper and lower faces of the edge; and each is written over its gamic.

If we draw $12_{,}$ to the mid-point of the edge 23, the face 3 becomes a 4-gon, the summit 3 a 4-ace, and we have the 4-gon 1 and the 4-ace 1. The 4_0 of the base and summit becomes 4_0 and $3_{0,}$, and the new edges *oo'* and its gamic appear; while instead of 23, the second edge and its gamic, we have $22_{,}$ and $2_{,}3$ with their gamics. The result is

$$\left.\begin{array}{llllll} 3_{0,}\mathbf{4}_1 3_4\mathbf{3}_2 & 3_{0,}\mathbf{3}_2 4_3\mathbf{3}_{2,} & 4_0\mathbf{3}_{2,} 4_3\mathbf{4}_3 & 4_0\mathbf{4}_3 3_2\mathbf{3}_4 & 4_0\mathbf{3}_4 4_1\mathbf{4}_1 & 3_{0,}\mathbf{3}_{2,} 4_0\mathbf{4}_1 \\ 4_1\mathbf{3}_4 3_2\mathbf{3}_{0,} & 3_2\mathbf{4}_3 3_{2,}\mathbf{3}_{0,} & 3_{2,}\mathbf{4}_3 4_3\mathbf{4}_0 & 4_3\mathbf{3}_2 3_4\mathbf{4}_0 & 3_4\mathbf{4}_1 4_1\mathbf{4}_0 & 3_{2,}\mathbf{4}_0 4_1\mathbf{3}_{0,} \end{array}\right\} \quad (12_{,})$$

The results of drawing $13_{,}$ to the mid-point of 34, $23_{,}$ to the middle of 34, and $24_{,}$ to the middle of 14, are in order following:—

$$\left.\begin{array}{llllll} 4_0\mathbf{4}_1 3_4\mathbf{4}_2 & 4_0\mathbf{4}_2 3_3\mathbf{3}_3 & 4_0\mathbf{3}_3 4_2\mathbf{3}_{3,} & 3_{0,}\mathbf{3}_{3,} 4_2\mathbf{3}_4 & 3_{0,}\mathbf{3}_4 4_1\mathbf{4}_1 & 3_{0,}\mathbf{4}_1 4_0\mathbf{3}_{3,} \\ 4_1\mathbf{3}_4 4_2\mathbf{4}_0 & 4_2\mathbf{3}_3 3_3\mathbf{4}_0 & 3_3\mathbf{4}_2 3_{3,}\mathbf{4}_0 & 3_{3,}\mathbf{4}_2 3_4\mathbf{3}_{0,} & 3_4\mathbf{4}_1 4_1\mathbf{3}_{0,} & 4_1\mathbf{4}_0 3_{3,}\mathbf{3}_{0,} \end{array}\right\} \quad (13_{,})$$

$$\left.\begin{array}{llllll} 4_0\mathbf{3}_1 3_4\mathbf{5}_2 & 3_{0,}\mathbf{5}_2 3_3\mathbf{3}_3 & 3_{0,}\mathbf{3}_3 5_2\mathbf{3}_{3,} & 4_0\mathbf{3}_{3,} 5_2\mathbf{3}_4 & 4_0\mathbf{3}_4 3_1\mathbf{3}_1 & 4_0\mathbf{5}_2 3_{0,}\mathbf{3}_{3,} \\ 3_1\mathbf{3}_4 5_2\mathbf{4}_0 & 5_2\mathbf{3}_3 3_3\mathbf{3}_{0,} & 3_3\mathbf{5}_2 3_{3,}\mathbf{3}_{0,} & 3_{3,}\mathbf{5}_2 3_4\mathbf{4}_0 & 3_4\mathbf{3}_1 3_1\mathbf{4}_0 & 5_2\mathbf{3}_{0,} 3_{3,}\mathbf{4}_0 \end{array}\right\} \quad (23_{,})$$

$$\left.\begin{array}{llllll} 3_{0,}\mathbf{4}_1 3_4\mathbf{4}_2 & 4_0\mathbf{4}_2 3_3\mathbf{3}_3 & 4_0\mathbf{3}_3 4_2\mathbf{3}_4 & 4_0\mathbf{3}_4 4_1\mathbf{3}_{4,} & 3_{0,}\mathbf{3}_{4,} 4_1\mathbf{4}_1 & 3_{0,}\mathbf{4}_2 4_0\mathbf{3}_{4,} \\ 4_1\mathbf{3}_4 4_2\mathbf{3}_{0,} & 4_2\mathbf{3}_3 3_3\mathbf{4}_0 & 3_3\mathbf{4}_2 3_4\mathbf{4}_0 & 3_4\mathbf{4}_1 3_{4,}\mathbf{4}_0 & 3_{4,}\mathbf{4}_1 4_1\mathbf{3}_{0,} & 4_2\mathbf{4}_0 3_{4,}\mathbf{3}_{0,} \end{array}\right\} \quad (24_{,})$$

In all these, the only convanescible edges, not through *o* or $o_{,}$, are $3_{2,}\mathbf{4}_3 4_3\mathbf{4}_0$ and $3_4\mathbf{4}_1 4_1\mathbf{4}_0$ in $(12_{,})$, either of which is in two 4-laterals, and the summits of neither are in two faces having any other common edge. Either therefore is convanescible; but with their present signatures they will neither of them vanish so as to give a 5-ace by the union of their summits; nor is either of them a generator introduced into the 5-gonal Ω, in our construction of the $(r+2)$-edra P. For they are both nodal gamics; the first meeting its gamic at 3, and the second at 1, while the generators of the class P never met their gamics. The first edge cannot vanish without the loss of two edges at the summit 3, so that its union with $2_{,}$ will give only a 4-ace: the like remark is to be made on the second. Yet nothing but the signatures prevents either of these lines from vanishing so as to yield a pentace. Now as these signatures are not essential to any polyedron, it is necessary, before we pronounce positively that the s before us $(12_{,})$ cannot be generated from the 6-edral pyramid, to satisfy ourselves whether, by any rearrangement of the signatures, the autopolar character being preserved, this vanescible edge ε cannot be made to stand as the gamic of one not meeting it.

If we exchange the signatures of the 4-aces 1 and 3, of the triaces $o_{,}$ and 2, and of the triaces 4 and $2_{,}$, we have the result following, still nodally autopolar:—

$$\begin{array}{cccccc} 3_2\mathbf{4}_1 3_{2,}\mathbf{3}_2 & 3_2\mathbf{3}_2 4_1\mathbf{3}_{2,} & 4_0\mathbf{3}_{2,}4_1\mathbf{4}_3 & 4_0\mathbf{4}_3 3_{0,}\mathbf{3}_4 & 4_0\mathbf{3}_4 4_3\mathbf{4}_1 & 3_2\mathbf{3}_{2,}4_0\mathbf{4}_1 \\ 4_3\mathbf{3}_4 3_{0,}\mathbf{3}_{0,} & 3_{0,}\mathbf{4}_3 3_4\mathbf{3}_{0,} & 3_4\mathbf{4}_3 4_1\mathbf{4}_0 & 4_1\mathbf{3}_2 3_{2,}\mathbf{4}_0 & 3_{2,}\mathbf{4}_1 4_3\mathbf{4}_0 & 3_4\mathbf{4}_0 4_3\mathbf{3}_{0,}; \end{array}$$

which is, placing the gamics one over the other,

$$\begin{array}{cccccc} 3_2\mathbf{4}_1 3_{2,}\mathbf{3}_2 & 4_3\mathbf{3}_4 3_{0,}\mathbf{3}_{0,} & 4_0\mathbf{4}_3 3_{0,}\mathbf{3}_4 & 4_0\mathbf{3}_4 4_3\mathbf{4}_1 & 4_0\mathbf{3}_{2,}4_1\mathbf{4}_3 & 3_2\mathbf{3}_{2,}4_0\mathbf{4}_1 \\ 4_1\mathbf{3}_{2,}3_2\mathbf{3}_2\cdot & 3_4\mathbf{3}_{0,}3_{0,}\mathbf{4}_3 & 4_3\mathbf{3}_{0,}3_4\mathbf{4}_0 & 3_4\mathbf{4}_3 4_1\mathbf{4}_0 & 3_{2,}\mathbf{4}_1 4_3\mathbf{4}_0 & 3_{2,}\mathbf{4}_0 4_1\mathbf{3}_2, \end{array}$$

the heptaedron (P) made from the 6-edral pyramid, which has the two convanescibles $3_4\mathbf{4}_3 4_1\mathbf{4}_0$ and $3_{2,}\mathbf{4}_1 4_3\mathbf{4}_0$.

It is thus proved that one $(r+3)$-edron s, namely, when $r=4$, must be rejected as reducible to the $(r+2=)$6-edral pyramid, by the convanescence of an edge not in o. Wherefore $-0^{(r-4)^2}$ is to be added to the number found in XIX.

XXI. Next, let o' be the x-ace in ε. As the united rank of o and $o'=r+3$, $o' \ngtr \frac{1}{2}(r+3)$; then $y \nless \frac{1}{2}(r+3)$. Now $y \ngtr 5$, and can be $=5$ only when there is no convanescible ε, as it has been just proved: therefore $y \ngtr 4$ and $r \ngtr 5$. Then o' is either a 4-ace or a triace, and a 4-ace only when $o=o'$, in which case it is indifferent which of the two is called o. That is, o must be this x-ace in ε wherever $x>3$. We are then to look in our $(r+3)$-edron s for a convanescible edge through o. This can be none other than the common edge of the only 4-laterals, H and E, which are contiguous only when $\mathrm{H}=\mathrm{E}\pm 1$. The y-ace in ε cannot be a summit of O, because ε, being in the two faces O and E that have the edge OE, could not be convanescible. Therefore the y-ace is neither e nor $f_{,}$, these being both in O, but is another summit of O$'$, which must be a triace. And as $x+y=$the united rank of o and o', of which o is the x-ace, o' is a triace, and O$'$ is a triangle. Therefore ε is the edge $o(e\pm 1)$ through $(e\pm 1)$ a summit of the triangle O$'$, *i. e.* through (OEH), *i. e.* through $\big(\mathrm{OE}(\mathrm{E}\mp 1)\big)$, ε being $\mathrm{E}(\mathrm{E}\mp 1)$. Therefore O$'$E and O$'$(E$\mp$1) are edges of O$'$. The signatures of the pyramid about e are thus exhibited:

$$\begin{array}{ccccc} \ldots\mathrm{G}-2 & \mathrm{G}-1 & \mathrm{G} & \mathrm{G}+1\ldots \\ \ldots\ :\ e+1 & & e\ \ e-1 & :\ \ldots \end{array}$$

the colon being the point $f_{,}$. The edges of $(e, e+1, f_{,})$ are OO$'$, O$'$(G$-$1) and O$'$(G$-$2), the two last being O$'$E and O$'$(E$+$1); therefore G$-$1$=$E, and e is nodal. Two edges of $(e, e-1, f_{\prime})$ are O′G and O′(G+1), which are O′E and O′(E−1), wherefore G=e and e is nodal. In the first case, ε or (G−1)(G−2) is between two 4-laterals and is convanescible; in the second, ε or G(G+1) is for a like reason convanescible, so that its summits, $(e\pm1)$ and the r-ace o, will reduce s to a $(r+2)$-edron with a $(r+1)$-ace, *i. e.* to a pyramid. Consequently s, having the triangle $(e, e\pm1, f_{\prime})$ when e is nodal, and EO′ is an edge of s, is to be rejected when $r>3$, because of the vanescible pair O(E±1) and $o(e\pm1)$. And this is the only s to be rejected in our enumeration for any value of r because of such a pair. Adding to this the one above rejected for another reason, for $r=4$, we have $1+0^{(r-4)^2}$ to deduct from the $\frac{1}{2}\{r.(r-2)+2_{r-1}\}$ $(r+3)$-edra (S), for every value of r, on the account of being generable from the $(r+2)$-edral pyramid. This makes the number of (S) thus far ascertained to be

$$\tfrac{1}{2}\{r.(r-2)-2+2_{r-1}\}-0^{(r-4)^2}.$$

XXII. It is necessary that we inquire how many of these are repetitions of each other, or of those enumerated in the class (Q).

The leading system of every S is the two edges $o\eta$ and oo' in the triangle $o'o\eta$; $o\eta$ being the diagonal in the face F, from the summit $\eta=F_r$ or F_l, as the case may be. This $o\eta$ is evanescible, and by evanishing, makes oo' convanescible, so that the union of its summits gives the pyramidal summit ω. Of course the gamics of $o\eta'$ and oo' (OH′ and OO′) are the first convanescible, and the second, thereby made evanescible.

We are then first to examine whether any of these (S) have a second *triangle* $\alpha\beta\gamma$, *containing a leading system like that in* $oo'\eta$, whereby S can be reduced to a pyramid, on some r-gonal base different from Ω; and then to decide whether this S has been twice enumerated above.

Let $ef_{\prime}$ be the generator OO′, $f_{\prime}$ being the point $f\pm\frac{1}{2}$. Let the summit F_r (or F_l) be η; then the triangle $oo'\eta$ contains the evanescible $o\eta$ (or $o'\eta$) and the convanescible oo'. Let $r>5$ for the present: then, if o be not the simplest of the two summits oo', there is no pair of summits in S excluding o, whose united rank $=r+3$. For in general, η and e, the only summits not triaces, except o and o', are tessaraces; and if η is e, it is a pentace containing $ef_{\prime}$, two sides of the triangle $oo'\eta$, and two edges of Ω. That is, two tessaraces, or a pentace and a triace, are the amplest pair in S, not o' or o; whose united rank makes only 8, $<r+3$.

Now if there be a triangle $\alpha\beta\gamma$ containing a leading system different from that in $oo'\eta$, α the greatest of its summits must be o, because $\alpha+\beta$ gives the same sum with o and o' by a convanescing edge. This triangle is therefore $o\beta\gamma$. And as $o\gamma$ is evanescible, γ, being neither o' nor η, must be e, the only summit besides these not a triace; therefore this triangle is $oe\beta$: and β being not o', nor η nor e, is a triace; wherefore, since $o+\beta=o+o'$, o' is a triace, and O′ is the triangle $e(e\pm1)f_{\prime}$.

Again, β cannot be a summit of O, for oo' is an edge of E, because e is in OO′; and OE is an edge of S, because oe is. If, then, β were in O, $o\beta$ could not become convanescible by the evanescence of oe, because its summits $o\beta$ are in O and E collateral in

OE. Therefore β is in O′, and is $e \pm 1$. Now $o(e \pm 1)$ being convanescible after the evanescence of oe, must before that be in a face not triangular. Of such faces S has only E, O, and H the polar of η. But oo' is in E; therefore $o(e \pm 1)$ is in H.

The conditions, therefore, that the triangle $oe(e \pm 1)$ shall contain a second leading system, are, that O′ shall be the triangle $e(e \pm 1)f_{\prime}$ having the triaces $e \pm 1$ and $f_{\prime}$, that oe shall be an edge distinct from $o\eta$, and that $o(e \pm 1)$ shall be an edge of H, no triangle.

XXIII. The signatures of the pyramid about e are

$$\begin{matrix} \ldots(G-2) & & (G-1)\ G & & (G+1)\ldots \\ \ldots\quad : & (e+1) & & e\ \ (e-1) & :\quad \ldots \end{matrix}$$

where the colon is the point $f_{\prime}$. As the only summits of the figure S not in the contour of Ω are o and o', e has the edge oe, or $o'e$, or both. The edges of $(e, e+1, f)$ (the triangle O′) are O′(G−1) and O′(G−2), and of $(e, e-1, f_{\prime})$ are O′G and O′(G+1); of which none is O′E unless G−1=E, or G−2=E; or G=E, or G+1=E.

If the first or third be true, e is nodal.

If G−2=E, G−1=E+1, and $e+1$ is nodal.

If G+1=E, G=E−1, and $e-1$ is nodal.

Therefore O′E is never an edge of $(e, e \pm 1, f_{\prime})$, unless one summit of this triangle is nodal; and OE is always an edge of S, if no summit of O′ is nodal.

When no summit of O′ is nodal, $o(e \pm 1)$ is always an edge of a quadrilateral. For in the triangle $(e, e+1, f_{\prime})$ neither O′(G−1) nor O′(G−2) is O′(E+1), because $e+1$ not being nodal, E+1 is neither G−1 nor G−2. As then O′(E+1) is no edge, $o'(e+1)$ is none, and $o(e+1)$ is in G−2, whose summits are $e+1, f_{\prime}, e+2, o$.

And in the triangle $(e, e-1, f_{\prime})$ neither O′G nor O′(G+1) is O′(E−1); wherefore $o(e-1)$ is in G+1, whose summits are $e-1, f_{\prime}, e-2, o$.

Consequently, when no summit of the triangle O′ is nodal, S has two leading systems, and may be a repetition of some other of the $\frac{1}{2}\{r.(r-2)-2+2_{r-1}\}$ S above (XXI.) constructed.

XXIV. Let e be nodal, and G=E. The triangle $(e, e-1, f_{\prime})$ has the edge O′E; but the triangle $(e, e+1, f_{\prime})$ has not the edge O′E. In the latter case OE is an edge of S; but not in the former, unless e be a pentace containing ef, eo and eo', with two sides of Ω. But G=E has only the summits e, $e-1$, o, o'; therefore e is no pentace, and OE is not an edge of S in the former case. Hence when e is nodal, and G=E, one line $ef_{\prime}$ can be drawn to make one triangle O′ or $(e, e-1, f_{\prime})$ such that O′E is an edge of S, and OE is not; and another, $(e, e+1, f_{\prime})$ such that OE is an edge and O′E is not.

When O′E is no edge, O′(G−1) and O′(G−2) are neither of them O′(E+1) in $(e, e+1, f_{\prime})$; and neither O′G nor O′(G+1) is O′(E−1) in $(e, e-1, f_{\prime})$. Therefore O(E+1) is an edge in one case, and O(E−1) in the other; and $o(e+1)$ is in (G−2), whose summits are $e+1, f_{\prime}\ e+2, o$ in the former case, and $o(e-1)$ is in G+1, whose summits are $e-1, f_{\prime}, e-2, o$, in the latter. Hence $e \pm 1$ is a triace in O′.

The like conclusion follows from supposing e nodal, and G−1=E, that one generator

$ef_{\prime}$ can be drawn such that OE and not O′E shall be an edge of S, and another $ef_{\prime}$ such that O′E and not OE shall be an edge. And when O′E is no edge, $o(e\pm1)$ is an edge of a quadrilateral. And $(e\pm1)$ is a triace in O′.

Thus whenever e is nodal in $(e, e\pm1, f_{\prime})$ and O′E is no edge, S may be a repetition of some other of the (S) above constructed, having the triangle $(o, e\pm1, e)$ which contains a second leading system.

Let now $e+1$ be nodal in the triangle $(e, e+1, f_{\prime})$, and let $e+1=G-2$. Here $e+1$ is a tessarace, because $G-2$ is a quadrilateral $(e+1, f_{\prime}, e+2, o')$; for $o'(e+1)$ is an edge because $O'(G-2)$ is. In this case therefore O′ has two tessaraces, and there is no second leading system in S.

Let then $e+1$ be nodal and $e+1=G-1$, or $e=G-2$. Here e is a pentace; for E has the point $f_{\prime}$. In this case oe is $o\eta$, and there is no second leading system.

In like manner it can be proved that if $(e-1)$ is nodal in the triangle $(e, e-1, f_{\prime})$, either $e-1$ is a tessarace, as well as e, in O′, or that e is the pentace η.

Consequently S is never repeated when O′ has a nodal summit, except when e is nodal, and O′E is no edge of S.

Thus we are tempted to conclude that S is always twice counted among those of the $\frac{1}{2}\{r.(r-2)-2+2_{r-1}\}$ which have a triangular O′, except when $(e\pm1)$ in O′ is nodal, or when e is nodal and O′E is an edge.

XXVI. There are two triangles O′ in which $ef_{\prime}$ is drawn from a nodal summit e to make an edge O′E, i. e. e may be either of the nodal summits of the pyramid, but as $ef_{\prime}$ drawn thus from one nodal summit is the fellow-generator in Ω of the other, only one of the resulting S was enumerated in XIX. And this is the s rejected in XXI. Also there are two lines $ef_{\prime}$ which can be drawn about either nodal summit to cut off a triangle $(e, e\pm1, f_{\prime})$ having $(e\pm1)$ nodal; but the two lines so drawn about one nodal summit of Ω are fellow-generators of those so drawn about the other. We have therefore among the $\frac{1}{2}\{r(r-2)+2_{r-1}\}$ counted in XIX., three of those *which have a triangular* O′, which have not two leading systems: and we have not more than three.

XXVII. We have now to determine how many of these (S) which have a triangle O′ and two leading systems are repetitions of each other.

Let S′ be one of these. Its base O is r-gonal, having all the summits of Ω except $(e\pm1)$, and instead of this the summit $f_{\prime}$. By the vanishing of $o\eta$ and $oo'_{\prime}$ the pyramid is restored, having the r-gonal base

$$\ldots(e+2)(e+1)e(e-1)(e-2)\ldots(g+1)g(g-1)(g-2)\ldots \quad (\Omega)$$

If O′ is $(e, e-1, f_{\prime})$, the r-gon O is

$$\ldots(e+2)(e+1)e.f_{\prime}(e-2)\ldots(g+1)g(g-1)(g-2)\ldots \quad (\text{O})$$

If O′ is $(e, e+1, f_{\prime})$, the r-gon O is

$$\ldots(e+2)f_{\prime}e(e-1)(e-2)\ldots(g+1)g(g-1)(g-2)\ldots \quad (\text{O})_{\prime}$$

When O is (O), S′ has the triace $(e-1)$, which is $(G, G+1, O')$, and has also the triangle

$(g, g+1, o')$, and (O) has all the summits of S′ except o, o' and $e-1$. By the vanishing of one of the leading systems of this S′, S′ becomes the pyramid, and (O) becomes (Ω); losing the triace f_i and receiving instead the triace $(e-1)$. By the vanishing of the other leading system, the same S′ becomes a pyramid on the r-gonal base (Ω)′,

$$..(e+2)(e+1)e.f_i(e-2)...(g+1)o'(g-1)(g-2).... \quad . \quad . \quad (\Omega)'$$

losing the triace g, and receiving instead the triace o'. Observe that $g+1$ is a tessarace and g a triace, as is evident from their polars G and G+1, by inspection of

$$\begin{array}{c} ...G-2,\ G-1,\ G,\ G+1... \\ :e+1,\ e\ \ e-1: \end{array}$$

where G is $\big(o\ e(e-1)\big)$ and G+1 is $\big(o(e-1)f_i(e-2)\big)$.

When O is $(O)_{i}$, $(O)_i$ becomes (Ω) by losing the triace f_i and receiving in its stead the triace $(e+1)$. Or it may become a pyramid on the r-gonal base

$$..(e+2)f_i\ e(e-1)(e-2)...(g+1)g\ o'(g-2)... \quad . \quad . \quad (\Omega)''$$

by the vanishing of its second leading system, receiving the triace o' and losing the triace $(g-1)$.

It is evident that the pyramids whose bases are Ω, (Ω)′ and (Ω)″ are identical, having the same r signatures with two slight changes of name, and the same nodal line through the summit 1. Now S′ having the triangle $\big(e(e-1)f'\big)$ is generated by drawing in the base (Ω)′ either the generator $e(e-1)$ from e to a point between f_i and $(e-2)$, or the generator $(g+1)g$ from $g+1$ to a point between o' and $g-1$. And S′ having the triangle $\big(e(e+1)f_i\big)$ is obtained by drawing in (Ω)″ either $e(e+1)$ from e to a point between f_i and $(e+2)$, or the generator $(g-2)(g-1)$ from $(g-2)$ to a point between o' and g.

XXVIII. Whenever these two generators drawn in (Ω)′, or (Ω)″, are not fellows, we have constructed S′ twice and twice counted it; but when they are fellows, we have constructed and counted it only once; for (XIX.) we have never used two fellow-generators in forming the $(r+3)$-edra S.

To examine this, let first r be odd. If fellow-generators can be drawn from e and $(g+1)$ in (Ω)′, we must have, since e is G_i in

$$\begin{array}{c} ...\quad (G-1)\ G \quad G(+1), \\ ...(e+1) \quad e\ (e-1) \quad : \end{array}$$

$$2G_i+2(G+1)=r+3+ \quad (VI.)\ (X.),$$

or

$$4-2G+2G+2=r+3+,$$

whence

$$3 \qquad =r,$$

or the pyramid is a tetraedron, contrary to hypothesis. And if fellow-generators can be drawn from e and $g-2$ in (Ω)″, we have, since e is G_i in

$$\begin{array}{c} ..(G-2) \quad (G-1)\ G.. \\ ..\quad :\quad (e+1) \quad e\ .., \end{array}$$

$$2G_i+2(G-2)=r+3,+$$

or

$$4-2G+2G-2=r+3+,$$

whence

$$-1 \qquad =r+,$$

which is still more absurd.

Therefore S′ is always made when r is odd from two different generators, and has been twice constructed among the $(r+3)$-edra S.

Let now r be even. If fellow-generators can be drawn in $(\Omega)'$ from e and $g+1$, we have

$$G_{\iota}=G+1\pm\tfrac{1}{2}r,$$

or

$$2-G=G+1\pm\tfrac{1}{2}r,$$

whence

$$\tfrac{1}{2}r=2G-1,$$

so that

$$r=4m-2.$$

If fellow-generators can be drawn from e and $g-2$ in $(\Omega)''$, we have (VIII.),

$$G_{\iota}=G-2\pm\tfrac{1}{2}r,$$

or

$$2-G=G-2\pm\tfrac{1}{2}r,$$

so that

$$r=4m$$

and

$$g-2=\frac{r}{4},$$

e being the opposite end of the diameter through $\tfrac{1}{4}r$.

It is thus proved that a single S′ having two leading systems is made by fellow-generators only, when r is even, and has been constructed and counted only once among the $(r+3)$-edra S. Every other S, having two leading systems, has been twice constructed from the even-angled Ω.

XXIX. The preceding reasoning from XXII. proceeds on the hypothesis that $r>5$. This restriction was made in order the more readily to prove that one summit of the triangle $\alpha\beta\gamma$ must be o. The same necessity is proved for $r=5$ thus.

In this triangle $\alpha\beta\gamma$ there is an evanescible edge, which is not $o\eta$, but equal in the united rank of its summits with $o\eta$. The only summits not triaces are the 4-aces e and η, and either two 4-aces o and o', or the pentace o.

The evanescible edge must either be oe or $e\eta$. Now when $e\eta$ is an edge $\eta=e\pm1$; that is H, the face in which f_{ι} is $=e\pm1$. We see from the signatures

$$\mathbf{1_5 2_4 3_3 4_2 5_1},$$

that either

$$H=3 \text{ and } e=2, \text{ or } H=2 \text{ and } e=1,$$

or else

$$H=2 \text{ and } e=3, \text{ or } H=1 \text{ and } e=2,$$

in all which cases O is a pentagon and O′ a triangle. Now no pentagon but O exists in the figure; therefore no triangle $\alpha\beta\gamma$ of which o is not a summit has an edge $\alpha\gamma$, the united rank of whose summits equals $5+4$, the united rank of o and η in the triangle $oo'\eta$. Therefore $\alpha=o$ also when $r=5$, and all the reasoning from article XX. is applicable to that value of r.

When $r=4$, we see from

$$1_4 2_3 3_2 4_1$$

that $e\eta$ is not an edge, unless either

$$H=2 \text{ and } e=1,$$

or

$$H=4 \text{ and } e=3,$$

two equivalent conditions, either of which gives the heptaedron $(12_{/})$ of art. XX., which is to be rejected on account of its being reducible to the 6-edral pyramid. The only other system, $H=2=e$, and $H=4=e$, both alike, is the S′ having two leading systems, made with fellow-generators, m being $=1$ in the formula $r=4m$. No other has two leading systems. Therefore all the reasoning of articles XXII....XXVIII. applies equally to every value of $r>3$.

The number of $(r+3)$-edra (S′) having a triangular O′ which we constructed in XIX. is exactly r, namely, half the number of generators λ, which all pair themselves into fellow-generators $e(e\pm\frac{3}{2})$. Of these we have proved that there are always three which have only a single leading system; and it has just been shown that, when r is even, there is always a fourth, having two leading but fellow systems. The number to be deducted from our enumeration on the score of their being repetitions of some other S′, is therefore

$$\tfrac{1}{2}(r-3-2_r)$$

(XXI.), to be subtracted from

$$\tfrac{1}{2}\left(r.(r-2)-2+2_{r-1}\right)-0^{(r-4)^2}.$$

The remainder is

$$\tfrac{1}{2}(r^2-3r+2)-0^{(r-4)^2}=\Pi''_{//},$$

the number of $(r+3)$-edra (S) thus far known, which is to be added to $\Pi_{//}+\Pi'_{//}$ in XVIII.

XXX. It remains that we inquire how many of these (S) are repetitions of those (Q) enumerated in XVI. If Q′ one of this class (Q) is one of (S), it has an evanescible edge the united rank of whose summits is $r+3$. The highest rank of o o' or o'' in Q′, the summits into which the vertex of the pyramid was broken, is $r-2$; for the united rank of the three $\not> r+4$, and the smallest have together at least six edges. There must therefore be in Q′ a pentace distinct from o o' and o''; for $r+3-(r-2)=5$. Now Q has this pentace only when the generators ef and eh are drawn from one summit e. Then this evanescible edge, if it passes through one of o o' o'', the greatest of which is o, must be oe, and o' and o'' must be triaces. Again, if oe in Q′ be the evanescible edge of the leading system of S, there will be a convanescible edge through either o or e, which vanishing after oe, will make o or e an r-ace. The only summits not triaces in Q′ are o, e, h, and f; and it is impossible that the summit o after the loss of the edge oe, can unite with either h or f, which are both 4-aces, to make an r-ace; for $r-3+4<r+2$.

Therefore the convanescible edge of the S-system of Q′ must be $e(e\pm1)$. But $e\pm1\not>3$;

so that $r+3 \not> 5+3$, and $r \not> 5$. Also $r \not< 5$, because Q$'$ cannot be generated, when $r<5$. We have then to endeavour to draw *ef* and *eh* in the base of the 6-edral pyramid, so that *ef* shall be evanescible, and $e(e\pm 1)$ then convanescible (for *eo* cannot vanish, since o o' and o'' are all triaces of necessity). It would be trifling with the reader to write a demonstration of what inspection of the pentagon proves in a moment, that of the three summits which e may be, 1, 2, and 5, 2 and 5 alone fulfil the conditions. Draw 25 and 24; then 25 is evanescible, and by its evanescence 21 becomes convanescible. Again, draw 53 and 52; then by the evanescence of 53, 54 becomes convanescible. These two Q have therefore been enumerated both in the class (Q) and the class (S), from which latter we have then to subtract two when $r=5$. That is, we must add $-2.0^{(r-5)^2}$, to $\Pi''_{\prime\prime}$ before found in XXIX. This makes up

$$\Pi''_{\prime\prime}=\tfrac{1}{2}(r^2-3r+2)-0^{(r-4)^2}-2.0^{(r-5)^2},$$

the correct number of the $(r+3)$-edra (S) generable from $(r+1)$-edral pyramid with nodal signatures. And thus the Problem (XIV.) is solved.

XXXI. It is yet required that we consider (VI.) the pyramid on an odd-angled base *with enodal signatures*, and determine whether, by operating upon it, we can obtain $(r+2)$-edra or $(r+3)$-edra which we have not already constructed with nodal signatures.

The nodal arrangement becomes enodal in Ω the base of the pyramid, if without disturbing those of the faces we exchange the signatures at the extremity of every nodal parallel, including among them that edge of Ω which is bisected by the *nodal axis* (VI.). After these exchanges every summit stands as the pole of the face opposite to it. And, conversely, an enodal signature of a pyramid becomes nodal, if all the diagonals parallel to any side of the enodal Ω be drawn, and then the signatures be exchanged both in that side and in these diagonals. The line bisecting that side, and passing through the opposite summit, becomes the *nodal axis*. All this will be clear to the reader, if he will have the goodness to draw a pentagon or a heptagon, and make these exchanges.

Let $r=2k+1$. All the single generators that can be drawn in the enodal Ω to generate a $(r+2)$-edral autopolar are among the parallels to the side $(k, k+1)$. Let $(e, r-e)$ be one of these: it is perpendicular to the axis of symmetry $(r, k+\frac{1}{2})$, which passes through r and bisects Ω. If E and (R$-$E) become 4-laterals, opposite to the 4-aces, e, and $(r-e)$, the edge $\big(\text{E}, (\text{R}-\text{E})\big)$ is the gamic of $\big(e, (r-e)\big)$, and the result is autopolar.

Let now the signatures be exchanged in $\big(e(r-e)\big)$ and on both sides the axis in lines parallel to $\big(e(r-e)\big)$. The result is still autopolar with nodal signatures; from which the nodally autopolar pyramid may be obtained by the evanescence of $(e, r-e)$ and the convanescence of (E, R$-$E); its nodal axis will be that axis of symmetry.

Thus it appears that every enodally autopolar $(r+2)$-edron generable from the enodal $(r+1)$-edral pyramid is also nodally autopolar, and generable from the same nodal pyramid. That is, it is one of the $(r+2)$-edra P already enumerated.

XXXII. Suppose that instead of drawing one diagonal $(e, r-e)$ perpendicular to the axis of symmetry, we had drawn any pair of diagonals so as to preserve symmetry, such as two $(e, r-e)$ and $(e_{,}, r+e_{,})$, or the pair (r, e) $(r, r-e)$, or the pair $(e, e+f)$, $(r-e, r-e-f)$; of which the first pair are parallels, the second pair meet in the pentace r, and the third pair are equidistant from the centre of the axis of symmetry on opposite sides of it; or suppose that we had drawn any number $2k$, of such lines, making k pairs, each preserving the symmetry about the axis through r. The gamic operations being performed in the faces opposite the affected summits, the symmetry would remain, and the result would be autopolar. If, next, the summits e and $(r-e)$, $(e+f)$ and $(r-e-f)$, &c., be exchanged in the parallels to $(k, k+1)$, the arrangement will become nodal and remain autopolar. The pair $(e, r-e)$ and $(e_{,}, r-e_{,})$ do not change their names; the pair (r, e) and $(r, r-e)$ make a simple exchange of names with each other, as do also the pair $(e, e+f)$ and $(r-e, r-e-f)$. The line (r, e) of the enodal arrangement, by changing its name to $(r, r-e)$ in the nodal arrangement, is no longer the gamic to (R, E); but to (R, R—E) an edge symmetrically placed with (R, E) about the axis of symmetry through r. From these considerations, which need not be further dwelt upon, it is plain that all the enodally autopolar $(r+3)$-edra, generable by symmetrically drawn pairs of generators in the enodal Ω, are also nodal, and are consequently among the class (Q) which have been already enumerated. For every one of them, in its nodal shape, can be reduced to the nodally-signed pyramid from which the (Q) were generated.

As any bisector of the enodal Ω from any summit is an axis of symmetry, it is clear that all the parallel or symmetric pairs of generators that can be drawn in Ω, are among those just considered about the axis through r. And, for the same reason, of any pair whatever that can be drawn in the enodal Ω, one may be always assumed to be drawn from r; or, if it be more convenient, perpendicular to the axis through r; and if two are drawn from one point, or to meet in a point, this point may be assumed to be r.

It may be hereafter convenient to define autopolarity as of three kinds, nodal, enodal, and utral; understanding by the first, purely nodal, or incapable of signature without two nodal summits, and by the second, purely enodal, or incapable of signature with a nodal summit or summits. The autopolars just reviewed, which are capable of both nodal and enodal signature, are then utrally autopolar.

XXXIII. Now let two non-parallel and non-symmetrical generators in the enodal Ω be drawn. We may assume one (r, e) to be drawn from r; the other is not parallel to (r, e), nor of the same length with it, whether it meets it at r or not. If this other be (r, f), whether f be on the same side or not of the axis of symmetry with (r, e), it is impossible that by exchanging the signatures of the diagonals perpendicular to that axis, the result should be autopolar. For e and f being 4-aces not in the same perpendicular, will after the exchanges be triaces, while E and F remain 4-laterals. And if (r, e) (k, l) be the two generators, whether (k, l) is or is not perpendicular to the axis of symmetry, e, k and l cannot all, after the exchanges, be 4-aces; but one of them (e or k) will appear as a triace, while E or K remains a 4-lateral. The enodal Q made by this (r, e) and (k, f), or by this (r, e) and (k, l), is then not capable of being made nodal, and thus reducible to the nodally-signed pyramid: it must therefore be a different $(r+3)$-edron from all the nodal (Q) before constructed. We may call it *purely enodal.*

We have then to add to the nodally autopolars (Q) all those enodally autopolars generable by a pair of diagonals of the enodal Ω, which are neither a parallel nor an equal pair. And we have to enumerate these pairs, taking care that no one is a repetition or a reflexion of another in the enodal Ω.

As any of the summits of Ω may be r, the number we are seeking cannot be greater than $\frac{1}{r}$th of all the possible pairs of not-crossing diagonals of Ω; that is, the function of r, V_r required, is not of a degree higher than the third. And it is evident that it must vanish, for $r=3$ and $r=5$, and must be

$$V_r=(r-3)(r-5)(ar+b).$$

By trial we find readily that

$$V_7=4.2.(7a+b)=2,$$

and

$$V_9=6.4.(9a+b)=8,$$

V_7 being the two pairs (72, 73) and (72, 74); and V_9 being (92, 93), (92, 94), (92, 95), (92, 96), (92, 84), (92, 85), (93, 94), (93, 95). Hence comes

$$V_r=\tfrac{1}{24}.(r-3)(r-5)(r-1).$$

And this is the number of *purely enodal* $(r+3)$-edra (Q), which are generable from the $(r+1)$-edral pyramid (when r is odd only). That is, we have to join to our previous enumeration the number

$$(Q)''=\tfrac{1}{24}.(r-3)(r-5)(r-1).2_{r-1},$$

to be added to $\Pi_{\prime\prime}+\Pi'_{\prime\prime}+\Pi''_{\prime\prime}$ of XVIII., XXIX., XXX.

XXXIV. We have yet to determine whether any purely enodal $(r+3)$-edra S can be generated by a line drawn from any summit to the mid-point of any edge of Ω.

When $r=4k+1$, the enodal signatures read thus:

$$\begin{matrix} \ldots \mathrm{R} & (\mathrm{R}-1), & & \ldots & & (\mathrm{R}-\mathrm{K}) & (\mathrm{R}-\mathrm{K}-1) & & \ldots \\ \ldots & 2k & & (2k-1)\ldots(k+1) & & k & : & k-1\ldots \end{matrix}$$

Let an axis of symmetry be drawn through r; and let the generator $(k+1, f_{\prime})$ be drawn from $(k+1)$ to the mid-point of $(k, k-1)$. $\big(\mathrm{R}-(\mathrm{K}+1)\big)$ and $(\mathrm{K}+1)$ are now quadrilaterals, and $\big(r-(k+1)\big)$ and $(k+1)$ are tessaraces, if the gamic operations be completed; and these tessaraces are upon the diagonal perpendicular to the axis of symmetry, being at equal distances from r. We have the triangles and triaces

$$\mathrm{O}'=(k+1, k, f_{\prime}), \qquad \mathrm{K}=(o', r-k, r-k-1),$$
$$o'=(\mathrm{K}+1, \mathrm{K}, \mathrm{F}_{\prime}), \qquad k=(\mathrm{O}', \mathrm{R}-\mathrm{K}, \mathrm{R}-\mathrm{K}-1),$$

$\mathrm{F}_{\prime}$ being the triangle polar to $f_{\prime}$, introduced between K and K−1. Leaving now the signatures of the faces undisturbed, let $r-x$ be exchanged for x, upon the diagonals perpendicular to the axis of symmetry, except that $r-k$ shall be exchanged for $f_{\prime}$; and let o' be exchanged for k. The result will be a nodally autopolar $(r+3)$-edron S, having the triangles and triaces

$$\mathrm{O}'=(r-k-1,\ o',\ r-k), \qquad \mathrm{K}=(k,\ f_{\prime},\ k+1),$$
$$k=(\mathrm{K}+1,\ \mathrm{K},\ \mathrm{F}_{\prime}), \qquad o'=(\mathrm{O}',\ \mathrm{R}-\mathrm{K},\ \mathrm{R}-\mathrm{K}-1),$$

in which o' and k are evidently nodal summits.

When $r=4k+3$, the enodal pyramid reads thus:

$$\ldots \mathrm{R} \quad (\mathrm{R}-1) \quad \ldots (\mathrm{R}-\mathrm{K}) \quad (\mathrm{R}-\mathrm{K}-1) \ldots$$
$$2k+1 \quad 2k \ldots \quad : \quad k+1 \quad k \ldots$$

The axis of symmetry being drawn through r, let the generator $(k, f_{\prime})$ be drawn from k to the mid-point of $(k+1,\ k+2)$. We have now the 4-laterals K and R$-$K, and the triaces k and $r-k$. The figure has the triangles and triaces

$$\mathrm{O}'=(k,\ k+1,\ f_{\prime}), \qquad \mathrm{K}+1=(o',\ r-k,\ r-k-1),$$
$$o'=(\mathrm{K},\ \mathrm{K}+1,\ \mathrm{F}_{\prime}), \qquad k+1=(\mathrm{O}',\ \mathrm{R}-\mathrm{K},\ \mathrm{R}-\mathrm{K}-1).$$

If we now exchange upon the diagonals perpendicular to the axis of symmetry $r-e$ for e, except only that $r-k-1$ is to be exchanged for $f_{\prime}$, and at the same time exchange o' and $k+1$, we have again a nodally autopolar (S), having the triangles and triaces

$$\mathrm{O}'=(r-k,\ o',\ r-k-1), \qquad \mathrm{K}+1=(k+1,\ k,\ f_{\prime}),$$
$$o'=(\mathrm{R}-\mathrm{K},\ \mathrm{O}',\ \mathrm{R}-\mathrm{K}-1), \qquad k+1=(\mathrm{K}+1,\ \mathrm{K},\ \mathrm{F}_{\prime});$$

in which o' and $k+1$ are the nodal summits.

Since all summits of the enodal Ω are alike, as origins of generators, no generators can be drawn to make an r-gonal O and a triangular O$'$ different from these; therefore no *purely enodal* $(r+3)$-edron can be constructed to have such an O and O$'$.

If O$'$=O, the generator $(e, f+\frac{1}{2})$ is the axis of symmetry through e, and all the summits are triaces, except the pentace e. In this case the resulting $(r+3)$-edron is instantly made nodally autopolar, by the exchange of signatures on each side of the axis, which becomes the nodal axis. The gentle reader will kindly assist my demonstrations in this new and somewhat intricate subject, by drawing a 7-gon and a 9-gon, and joining the angles in the required way to two included summits o and o'.

XXXV. But when O$'$ is not a triangle, nor of equal rank with O, it becomes impossible to give to the S generated by the line $(e, f+\frac{1}{2})$ a nodal arrangement; for there is no summit with which o' can be exchanged. If $f_{\prime}$ is a point of the face H, $f_{\prime}$ and η are tessaraces which may take each other's place, but o', whether tessarace or m-ace, stands alone and immoveable. Consequently, the resulting $(r+3)$-edron is of *purely enodal autopolarity*, and, with all the other enodal (S) having no triangular O$'$, is to be added to our enumeration. The number of different generators $(r, f+\frac{1}{2})$ is $\frac{1}{2}(r-5)$, f receiving every value from $f=2$ to $f=\frac{1}{2}(r-3)$: this $\frac{1}{2}(r-5)$, the number of purely enodal autopolars (S), having no r-gonal O, is to be added to $(\mathrm{Q})''$ in XXXIII., completing the list of purely enodal autopolars. And thus the question of (XXXI.) is determined.

We have exhausted all the methods of adding two faces and two summits to the $(r+1)$-edral pyramid. By adding two faces in the base Ω, and their two poles about the vertex ω, we formed the $(r+3)$-edra (Q). By adding a face and a summit in Ω, and their polars about ω, we formed the $(r+3)$-edra (S).

Collecting, now, our results from XIII., XVIII., XXIX., XXX., XXXIII., we find the numbers Π_1 of $(r+2)$-edra and Π_2 of $(r+3)$-edra, nodal, enodal, and utral, which are generable from the $(r+1)$-edral pyramid, to be the following:

$$\Pi_1=\tfrac{1}{4}\{r^2-3r+2.4_{r-2}+(r-3).2_{r-1}\};$$

$$\begin{aligned}\Pi_2=&\tfrac{1}{24}\{r^2(r-4)(r-2)\}2_r\\ &+\tfrac{1}{48}\{2r.(r-3)(r-4)(r+1)-3.(r-3)(r-1)+3.(r-3)(r-3)\}2_{r-1}\\ &+\tfrac{1}{2}(r^2-3r+2)+\tfrac{1}{24}(r-3)(r-5)(r-1).2_{r-1}+\tfrac{1}{2}(r-5).2_{r-1}-0^{(r-4)^2.(r-7)^2}-2.0^{(r-5)^2},\end{aligned}$$

or

$$\Pi_2=\tfrac{1}{24}\{(r^4-6r^3+20r^2-36r+24)2_r+(r-3)(r^3-2r^2+5r+8)2_{r-1}-0^{(r-4)^2(r-7)^2}-2.0^{(r-5)^2}\}.$$

It has been proved that these Π_1 and Π_2, and these only, are the autopolar $(r+2)$-edra and $(r+3)$-edra generable from the $(r+1)$-edral pyramid, that none of them is reducible by vanescible pairs to a higher pyramid, nor any one a repetition of another.

The problem of enumeration of the x-edra may, by a slight extension of the meaning of the word partition, be stated thus: *to find the* k-*partitions of a pyramid.* This depends on another: *to find the* k-*partitions of a polygon*; which is also thus: *to find the* k-*partitions of a pencil.* By the k-partitions of a p-gon, I mean the number of ways, none a repetition or reflexion of another, in which k lines can be drawn in a p-gon, none crossing another, so as to make the system of 1 face and p summits into a system of $h+1$ faces and $p+i$ summits $(h+i=k)$, the k lines being terminated either by summits of the p-gon, or by i points chosen either on its edges or within its area; with the understanding that at least three lines shall meet in each of the i points, two of which will always be a side and its segment, when i is chosen on a side, the segment counting among the k lines.

Whenever a p-gon and a p-ace are similarly k-partitioned, a certain number of autopolar $(p+k+1)$-edra are obtained by different ways of applying the p-ace to the p-gon; when they are dissimilarly k-partitioned, or when the p-gon is k-partitioned and the p-ace $(k\pm l)$-partitioned, a certain number of heteropolars will arise from different ways of applying the p-ace to the p-gon. The direct or reverse manner of applying the p-ace to the p-gon will give nodal or enodal autopolars.

The way seems now clearly indicated, and partly laid open to the solution of a geometrical problem, which, while it seems at first sight almost elementary, has lain for centuries before mathematicians unanswered. *The enumeration of the* x-*edra* is a question of partition, and the k-partitions of a polygon, a pencil and a pyramid, will probably be found still more inaccessible than our new and unamiable acquaintances, *the commensurable* k-*partitions of a line.*

XXXVI. It may be useful to give a list of the autopolar n-edra as far as $n=8$, which are not pyramids.

The only such autopolar hexaedron is obtained by drawing 13 in the 4-diagonal Ω, dividing it into the triangles 6 and 5. Its ten edges are

$$\begin{array}{ccccc} 4_1\mathbf{3}_4 3_2\mathbf{3}_6 & 3_2\mathbf{4}_3 4_3\mathbf{3}_6 & 4_3\mathbf{3}_2 3_4\mathbf{3}_5 & 3_4\mathbf{4}_1 4_1\mathbf{3}_5 & 3_6\mathbf{4}_3 3_5\mathbf{4}_1 \\ \mathbf{4}_1 3_4\mathbf{3}_2 3_6 & \mathbf{3}_2 4_3\mathbf{4}_3 3_6 & \mathbf{4}_3 3_2\mathbf{3}_4 3_5 & \mathbf{3}_4 4_1\mathbf{4}_1 3_5 & \mathbf{3}_6 4_3\mathbf{3}_5 4_1, \end{array}$$

where the heavier type denotes the faces, and the lighter the summits. The leading system are the final pair of gamics, of which the first is convanescible and the second evanescible.

There are three autopolar 7-edra of the class P (VII.), obtained by drawing the generators 13, 14 and 25 in the pentagonal base Ω, all of the class P. These are—

$$\left.\begin{array}{cccccc} 4_1\mathbf{3}_5 3_2\mathbf{3}_7 & 3_2\mathbf{3}_4 4_3\mathbf{3}_7 & 4_3\mathbf{4}_3 3_4\mathbf{4}_6 & 3_4\mathbf{3}_2 3_5\mathbf{4}_6 & 3_5\mathbf{4}_1 4_1\mathbf{4}_6 & 3_7\mathbf{4}_3 4_6\mathbf{4}_1 \\ \mathbf{4}_1 3_5\mathbf{3}_2 3_7 & \mathbf{3}_2 3_4\mathbf{4}_3 3_7 & \mathbf{4}_3 4_3\mathbf{3}_4 4_6 & \mathbf{3}_4 3_2\mathbf{3}_5 4_6 & \mathbf{3}_5 4_1\mathbf{4}_1 4_6 & \mathbf{3}_7 4_3\mathbf{4}_6 4_1 \end{array}\right\} (13)$$

$$\left.\begin{array}{cccccc} 4_1\mathbf{3}_5 3_2\mathbf{4}_6 & 3_2\mathbf{4}_4 3_3\mathbf{4}_6 & 3_3\mathbf{3}_3 4_4\mathbf{4}_6 & 4_4\mathbf{3}_2 3_5\mathbf{3}_7 & 3_5\mathbf{4}_1 4_1\mathbf{3}_7 & 4_6\mathbf{4}_4 3_7\mathbf{4}_1 \\ \mathbf{4}_1 3_5\mathbf{3}_2 4_6 & \mathbf{3}_2 4_4\mathbf{3}_3 4_6 & \mathbf{3}_3 3_3\mathbf{4}_4 4_6 & \mathbf{4}_4 3_2\mathbf{3}_5 3_7 & \mathbf{3}_5 4_1\mathbf{4}_1 3_7 & \mathbf{4}_6 4_4\mathbf{3}_7 4_1 \end{array}\right\} (14)$$

$$\left.\begin{array}{cccccc} 3_1\mathbf{4}_5 4_2\mathbf{3}_7 & 4_2\mathbf{3}_4 3_3\mathbf{4}_6 & 3_3\mathbf{3}_3 3_4\mathbf{4}_6 & 3_4\mathbf{4}_2 4_5\mathbf{4}_6 & 4_5\mathbf{3}_1 3_1\mathbf{3}_7 & 3_6\mathbf{4}_2 4_7\mathbf{4}_5 \\ \mathbf{3}_1 4_5\mathbf{4}_2 3_7 & \mathbf{4}_2 3_4\mathbf{3}_3 4_6 & \mathbf{3}_3 3_3\mathbf{3}_4 4_6 & \mathbf{3}_4 4_2\mathbf{4}_5 4_6 & \mathbf{4}_5 3_1\mathbf{3}_1 3_7 & \mathbf{3}_6 4_2\mathbf{4}_7 4_5 \end{array}\right\} (25)$$

the last two of which are in order the 7-edra $(13_{,})$ and $(12_{,})$ of art. XX. They have all three quadrilaterals and four triangles.

Two more of the class S are generated from the 5-edral pyramid by drawing 25 to the mid-point of 14, and 25 to that of 34; viz.

$$\left.\begin{array}{cccccc} 4_1\mathbf{3}_4 4_2\mathbf{3}_7 & 4_2\mathbf{3}_3 3_3\mathbf{4}_6 & 3_3\mathbf{4}_2 3_4\mathbf{4}_6 & 3_4\mathbf{4}_1 3_5\mathbf{4}_6 & 3_5\mathbf{4}_1 4_1\mathbf{3}_6 & 3_7\mathbf{4}_2 4_6\mathbf{3}_5 \\ \mathbf{4}_1 3_4\mathbf{4}_2 3_7 & \mathbf{4}_2 3_3\mathbf{3}_3 4_6 & \mathbf{3}_3 4_2\mathbf{3}_4 4_6 & \mathbf{3}_4 4_1\mathbf{3}_5 4_6 & \mathbf{3}_5 4_1\mathbf{4}_1 3_6 & \mathbf{3}_7 4_2\mathbf{4}_6 3_5 \end{array}\right\} (2, 4\tfrac{1}{2})$$

$$\left.\begin{array}{cccccc} 3_1\mathbf{3}_4 5_2\mathbf{4}_6 & 5_2\mathbf{3}_3 3_3\mathbf{3}_7 & 3_3\mathbf{5}_2 3_5\mathbf{3}_7 & 3_5\mathbf{5}_2 3_4\mathbf{4}_6 & 3_4\mathbf{3}_1 3_1\mathbf{4}_6 & 4_6\mathbf{5}_2 3_7\mathbf{3}_5 \\ \mathbf{3}_1 3_4\mathbf{5}_2 4_6 & \mathbf{5}_2 3_3\mathbf{3}_3 3_7 & \mathbf{3}_3 5_2\mathbf{3}_5 3_7 & \mathbf{3}_5 5_2\mathbf{3}_4 4_6 & \mathbf{3}_4 3_1\mathbf{3}_1 4_6 & \mathbf{4}_6 5_2\mathbf{3}_7 3_5 \end{array}\right\} (2, 3\tfrac{1}{2})$$

Of these two, which are identical with $(23')$ and $(24')$ of art. XX., the first has three quadrilaterals and four triangles, while the second has a pentagon, a quadrilateral, and four triangles. In all these 7-edra 6 and 7 are the faces into which the base Ω is divided.

XXXVII. Of autopolar 8-edra there are five generable from the 7-edral pyramid, by drawing 13, 15, 35, 14, 36, in the 6-gonal Ω, all of the class P.

$4_1\mathbf{3}_6 3_2\mathbf{3}_8$	$3_2\mathbf{3}_5 4_3\mathbf{3}_8$	$4_3\mathbf{3}_4 3_4\mathbf{5}_7$	$3_4\mathbf{4}_3 3_5\mathbf{5}_7$	$3_5\mathbf{3}_2 3_6\mathbf{5}_7$	$3_6\mathbf{4}_1 4_1\mathbf{5}_7$	$5_7\mathbf{4}_1 3_8\mathbf{4}_3$	(13)
$\mathbf{4}_1 3_6\mathbf{3}_2 3_8$	$\mathbf{3}_2 3_5\mathbf{4}_3 3_8$	$\mathbf{4}_3 3_4\mathbf{3}_4 5_7$	$\mathbf{3}_4 4_3\mathbf{3}_5 5_7$	$\mathbf{3}_5 3_2\mathbf{3}_6 5_7$	$\mathbf{3}_6 4_1\mathbf{4}_1 5_7$	$\mathbf{5}_7 4_1\mathbf{3}_8 4_3$	

$4_1\mathbf{3}_6 3_2\mathbf{5}_7$	$3_2\mathbf{4}_5 3_3\mathbf{5}_7$	$3_3\mathbf{3}_4 3_4\mathbf{5}_7$	$3_4\mathbf{3}_3 4_5\mathbf{5}_7$	$4_5\mathbf{3}_2 3_6\mathbf{3}_8$	$3_6\mathbf{4}_1 4_1\mathbf{3}_8$	$3_8\mathbf{4}_1 5_7\mathbf{4}_5$	(15)
$\mathbf{4}_1 3_6\mathbf{3}_2 5_7$	$\mathbf{3}_2 4_5\mathbf{3}_3 5_7$	$\mathbf{3}_3 3_4\mathbf{3}_4 5_7$	$\mathbf{3}_4 3_3\mathbf{4}_5 5_7$	$\mathbf{4}_5 3_2\mathbf{3}_6 3_8$	$\mathbf{3}_6 4_1\mathbf{4}_1 3_8$	$\mathbf{3}_8 4_1\mathbf{5}_7 4_5$	

$3_1\mathbf{3}_6 3_2\mathbf{5}_7$	$3_2\mathbf{4}_5 4_3\mathbf{5}_7$	$4_3\mathbf{3}_4 3_4\mathbf{3}_8$	$3_4\mathbf{4}_3 4_5\mathbf{3}_8$	$4_5\mathbf{3}_2 3_6\mathbf{5}_7$	$3_6\mathbf{3}_1 3_1\mathbf{5}_7$	$5_7\mathbf{4}_3 3_8\mathbf{4}_5$	(35)
$\mathbf{3}_1 3_6\mathbf{3}_2 5_7$	$\mathbf{3}_2 4_5\mathbf{4}_3 5_7$	$\mathbf{4}_3 3_4\mathbf{3}_4 3_8$	$\mathbf{3}_4 4_3\mathbf{4}_5 3_8$	$\mathbf{4}_5 3_2\mathbf{3}_6 5_7$	$\mathbf{3}_6 3_1\mathbf{3}_1 5_7$	$\mathbf{5}_7 4_3\mathbf{3}_8 4_5$	

$4_1\mathbf{3}_6 3_2\mathbf{4}_8$	$3_2\mathbf{3}_5 3_3\mathbf{4}_8$	$3_3\mathbf{4}_4 4_4\mathbf{4}_8$	$4_4\mathbf{3}_3 3_5\mathbf{4}_7$	$3_5\mathbf{3}_2 3_6\mathbf{4}_7$	$3_6\mathbf{4}_1 4_1\mathbf{4}_7$	$4_7\mathbf{4}_1 4_8\mathbf{4}_4$	(14)
$\mathbf{4}_1 3_6\mathbf{3}_2 4_8$	$\mathbf{3}_2 3_5\mathbf{3}_3 4_8$	$\mathbf{3}_3 4_4\mathbf{4}_4 4_8$	$\mathbf{4}_4 3_3\mathbf{3}_5 4_7$	$\mathbf{3}_5 3_2\mathbf{3}_6 4_7$	$\mathbf{3}_6 4_1\mathbf{4}_1 4_7$	$\mathbf{4}_7 4_1\mathbf{4}_8 4_4$	

$3_1\mathbf{4}_6 3_2\mathbf{4}_7$	$3_2\mathbf{3}_5 4_3\mathbf{4}_7$	$4_3\mathbf{3}_4 3_4\mathbf{4}_8$	$3_4\mathbf{4}_3 3_5\mathbf{4}_8$	$3_5\mathbf{3}_2 4_6\mathbf{4}_8$	$4_6\mathbf{3}_1 3_1\mathbf{4}_7$	$4_7\mathbf{4}_3 4_8\mathbf{4}_6$	(36)
$\mathbf{3}_1 4_6\mathbf{3}_2 4_7$	$\mathbf{3}_2 3_5\mathbf{4}_3 4_7$	$\mathbf{4}_3 3_4\mathbf{3}_4 4_8$	$\mathbf{3}_4 4_3\mathbf{3}_5 4_8$	$\mathbf{3}_5 3_2\mathbf{4}_6 4_8$	$\mathbf{4}_6 3_1\mathbf{3}_1 4_7$	$\mathbf{4}_7 4_3\mathbf{4}_8 4_6$	

XXXVIII. There are also seven autopolar 8-edra generable from the 6-edral pyramid, viz. three of the class Q, generated by drawing 25 and 24, 13 and 14, 52 and 53, in the pentagon Ω, and four of the class S. The Q are these:—

$3_1\mathbf{4}_5 5_2\mathbf{3}_6$	$5_2\mathbf{4}_4 3_3\mathbf{3}_8$	$3_3\mathbf{3}_3 4_4\mathbf{3}_8$	$4_4\mathbf{5}_2 4_5\mathbf{3}_7$	$4_5\mathbf{3}_1 3_1\mathbf{3}_6$	$3_8\mathbf{4}_4 3_7\mathbf{5}_2$	$3_7\mathbf{4}_5 3_6\mathbf{5}_2$	(24, 25)
$\mathbf{3}_1 4_5\mathbf{5}_2 3_6$	$\mathbf{5}_2 4_4\mathbf{3}_3 3_8$	$\mathbf{3}_3 3_3\mathbf{4}_4 3_8$	$\mathbf{4}_4 5_2\mathbf{4}_5 3_7$	$\mathbf{4}_5 3_1\mathbf{3}_1 3_6$	$\mathbf{3}_8 4_4\mathbf{3}_7 5_2$	$\mathbf{3}_7 4_5\mathbf{3}_6 5_2$	

$5_1\mathbf{3}_5 3_2\mathbf{3}_6$	$3_2\mathbf{4}_4 4_3\mathbf{3}_6$	$4_3\mathbf{4}_3 4_4\mathbf{3}_7$	$4_4\mathbf{3}_2 3_5\mathbf{3}_8$	$3_5\mathbf{5}_1 5_1\mathbf{3}_8$	$3_8\mathbf{5}_1 3_7\mathbf{4}_4$	$3_7\mathbf{5}_1 3_6\mathbf{4}_3$	(13, 14)
$\mathbf{5}_1 3_5\mathbf{3}_2 3_6$	$\mathbf{3}_2 4_4\mathbf{4}_3 3_6$	$\mathbf{4}_3 4_3\mathbf{4}_4 3_7$	$\mathbf{4}_4 3_2\mathbf{3}_5 3_8$	$\mathbf{3}_5 5_1\mathbf{5}_1 3_8$	$\mathbf{3}_8 5_1\mathbf{3}_7 4_4$	$\mathbf{3}_7 5_1\mathbf{3}_6 4_3$	

$3_1\mathbf{5}_5 4_2\mathbf{3}_6$	$4_2\mathbf{3}_4 4_3\mathbf{3}_7$	$4_3\mathbf{4}_3 3_4\mathbf{3}_8$	$3_4\mathbf{4}_2 5_5\mathbf{3}_8$	$5_5\mathbf{3}_1 3_1\mathbf{3}_6$	$3_6\mathbf{4}_2 3_7\mathbf{5}_5$	$3_7\mathbf{4}_3 3_8\mathbf{5}_5$	(52, 53)
$\mathbf{3}_1 5_5\mathbf{4}_2 3_6$	$\mathbf{4}_2 3_4\mathbf{4}_3 3_7$	$\mathbf{4}_3 4_3\mathbf{3}_4 3_8$	$\mathbf{3}_4 4_2\mathbf{5}_5 3_8$	$\mathbf{5}_5 3_1\mathbf{3}_1 3_6$	$\mathbf{3}_6 4_2\mathbf{3}_7 5_5$	$\mathbf{3}_7 4_3\mathbf{3}_8 5_5$	

The autopolar 8-edra of the class S, which are distinct from the preceding, are the following four, obtained by the generators 16, 46, 56, all drawn to the mid-point of 23, and 16 drawn from 1 to the mid-point of 34, in the pentagon Ω.

$4_1\mathbf{3}_5 3_2\mathbf{3}_7$	$3_2\mathbf{4}_4 3_6\mathbf{3}_7$	$3_6\mathbf{4}_4 3_3\mathbf{5}_8$	$3_3\mathbf{3}_3 4_4\mathbf{5}_8$	$4_4\mathbf{3}_2 3_5\mathbf{5}_8$	$3_5\mathbf{4}_1 4_1\mathbf{5}_8$	$3_7\mathbf{3}_6 5_8\mathbf{4}_1$	$(1, 2\frac{1}{2})$
$\mathbf{4}_1 3_5\mathbf{3}_2 3_7$	$\mathbf{3}_2 4_4\mathbf{3}_6 3_7$	$\mathbf{3}_6 4_4\mathbf{3}_3 5_8$	$\mathbf{3}_3 3_3\mathbf{4}_4 5_8$	$\mathbf{4}_4 3_2\mathbf{3}_5 5_8$	$\mathbf{3}_5 4_1\mathbf{4}_1 5_8$	$\mathbf{3}_7 3_6\mathbf{5}_8 4_1$	

$3_1\mathbf{3}_5 3_2\mathbf{5}_8$	$3_2\mathbf{5}_4 3_6\mathbf{5}_8$	$3_6\mathbf{5}_4 3_3\mathbf{3}_7$	$3_3\mathbf{3}_3 5_4\mathbf{3}_7$	$5_4\mathbf{3}_2 3_5\mathbf{5}_8$	$3_5\mathbf{3}_1 3_1\mathbf{5}_8$	$5_8\mathbf{3}_6 3_7\mathbf{5}_4$	$(4, 2\frac{1}{2})$
$\mathbf{3}_1 3_5\mathbf{3}_2 5_8$	$\mathbf{3}_2 5_4\mathbf{3}_6 5_8$	$\mathbf{3}_6 5_4\mathbf{3}_3 3_7$	$\mathbf{3}_3 3_3\mathbf{5}_4 3_7$	$\mathbf{5}_4 3_2\mathbf{3}_5 5_8$	$\mathbf{3}_5 3_1\mathbf{3}_1 5_8$	$\mathbf{5}_8 3_6\mathbf{3}_7 5_4$	

$3_1\mathbf{4}_5 3_2\mathbf{4}_8$	$3_2\mathbf{4}_4 3_6\mathbf{4}_8$	$3_6\mathbf{4}_4 3_3\mathbf{4}_7$	$3_3\mathbf{3}_3 4_4\mathbf{4}_7$	$4_4\mathbf{3}_2 4_5\mathbf{4}_7$	$4_5\mathbf{3}_1 3_1\mathbf{4}_8$	$4_8\mathbf{3}_6 4_7\mathbf{4}_5$	$(5, 2\frac{1}{2})$
$\mathbf{3}_1 4_5\mathbf{3}_2 4_8$	$\mathbf{3}_2 4_4\mathbf{3}_6 4_8$	$\mathbf{3}_6 4_4\mathbf{3}_3 4_7$	$\mathbf{3}_3 3_3\mathbf{4}_4 4_7$	$\mathbf{4}_4 3_2\mathbf{4}_5 4_7$	$\mathbf{4}_5 3_1\mathbf{3}_1 4_8$	$\mathbf{4}_8 3_6\mathbf{4}_7 4_5$	

$4_1\mathbf{3}_5 3_2\mathbf{4}_8$	$3_2\mathbf{3}_4 4_3\mathbf{4}_8$	$4_3\mathbf{4}_3 3_6\mathbf{4}_8$	$3_6\mathbf{4}_3 3_4\mathbf{4}_7$	$3_4\mathbf{3}_2 3_5\mathbf{4}_7$	$3_5\mathbf{4}_1 4_1\mathbf{4}_7$	$4_7\mathbf{4}_1 4_8\mathbf{3}_6$	$(1, 3\frac{1}{2})$
$\mathbf{4}_1 3_5\mathbf{3}_2 4_8$	$\mathbf{3}_2 3_4\mathbf{4}_3 4_8$	$\mathbf{4}_3 4_3\mathbf{3}_6 4_8$	$\mathbf{3}_6 4_3\mathbf{3}_4 4_7$	$\mathbf{3}_4 3_2\mathbf{3}_5 4_7$	$\mathbf{3}_5 4_1\mathbf{4}_1 4_7$	$\mathbf{4}_7 4_1\mathbf{4}_8 3_6$	

In all these 7 and 8 are the faces into which Ω is divided.

There are yet three autopolar 8-edra not included in the classes discussed in this memoir, being $(r+4)$-edra generable by gamic pairs from the $(r+1)$-edral pyramid. The first is obtained by drawing in the 4-gonal Ω, 42, and then 45 to the middle of 12; the second by drawing 24 and then 38 to the middle of 24; the third by joining 65, in the mid-points of 14 and 12, as follows:—

$$\left.\begin{matrix} 3_1\mathbf{6}_4 3_5\mathbf{3}_6 & 3_5\mathbf{6}_4 4_2\mathbf{3}_7 & 4_2\mathbf{3}_3 3_3\mathbf{3}_8 & 3_3\mathbf{4}_2 6_4\mathbf{3}_8 & 6_4\mathbf{3}_1 3_1\mathbf{3}_6 & 3_6\mathbf{3}_5 3_7\mathbf{6}_4 & 3_7\mathbf{4}_2 3_8\mathbf{6}_4 \\ \mathbf{3}_1 6_4\mathbf{3}_5 3_6 & \mathbf{3}_5 6_4\mathbf{4}_2 3_7 & \mathbf{4}_2 3_3\mathbf{3}_3 3_8 & \mathbf{3}_3 4_2\mathbf{6}_4 3_8 & \mathbf{6}_4 3_1\mathbf{3}_1 3_6 & \mathbf{3}_6 3_5\mathbf{3}_7 6_4 & \mathbf{3}_7 4_2\mathbf{3}_8 6_4 \end{matrix}\right\}(42, 41\tfrac{1}{2})$$

$$\left.\begin{matrix} 3_1\mathbf{4}_4 4_2\mathbf{4}_5 & 4_2\mathbf{4}_3 4_3\mathbf{3}_6 & 4_3\mathbf{4}_2 4_4\mathbf{3}_7 & 3_6\mathbf{4}_3 3_7\mathbf{3}_8 & 3_7\mathbf{4}_4 4_5\mathbf{3}_8 & 4_5\mathbf{4}_2 3_6\mathbf{3}_8 & 4_4\mathbf{3}_1 3_1\mathbf{4}_5 \\ \mathbf{3}_1 4_4\mathbf{4}_2 4_5 & \mathbf{4}_2 4_3\mathbf{4}_3 3_6 & \mathbf{4}_3 4_2\mathbf{4}_4 3_7 & \mathbf{3}_6 4_3\mathbf{3}_7 3_8 & \mathbf{3}_7 4_4\mathbf{4}_5 3_8 & \mathbf{4}_5 4_2\mathbf{3}_6 3_8 & \mathbf{4}_4 3_1\mathbf{3}_1 4_5 \end{matrix}\right\}(83, 82, 84)$$

$$\left.\begin{matrix} 4_1\mathbf{4}_4 3_5\mathbf{3}_7 & 3_5\mathbf{4}_4 3_2\mathbf{5}_8 & 3_2\mathbf{3}_3 3_3\mathbf{5}_8 & 3_3\mathbf{3}_2 4_4\mathbf{5}_8 & 4_4\mathbf{4}_1 3_6\mathbf{5}_8 & 3_6\mathbf{4}_1 4_1\mathbf{3}_7 & 3_7\mathbf{3}_5 5_8\mathbf{3}_6 \\ \mathbf{4}_1 4_4\mathbf{3}_5 3_7 & \mathbf{3}_5 4_4\mathbf{3}_2 5_8 & \mathbf{3}_2 3_3\mathbf{3}_3 5_8 & \mathbf{3}_3 3_2\mathbf{4}_4 5_8 & \mathbf{4}_4 4_1\mathbf{3}_6 5_8 & \mathbf{3}_6 4_1\mathbf{4}_1 3_7 & \mathbf{3}_7 3_5\mathbf{5}_8 3_6 \end{matrix}\right\}(1\tfrac{1}{2}, 4\tfrac{1}{2})$$

Thus we find that there are one autopolar 6-edron, five autopolar 7-edra, and fifteen autopolar 8-edra, besides the three pyramids.

XXXIX. We may, for example, effect the reduction of the three last written 8-edra to the 5-edral pyramid. The only vanescible pair in the first is the seventh pair. As it stands, the summits and their signatures read thus,

$$3_1 4_2 3_3 6_4 3_5 3_6 3_7 3_8 :$$

after the disappearance of that pair, which unites the seventh and eighth summits and degrades the second and fourth, they read

$$3_1 3_2 3_3 5_4 3_5 3_6 4_{7\,8}.$$

The second pair are now vanescible and next vanish, and we read thus,

$$3_1 4_{2\,5} 3_3 4_4 3_6 3_{7\,8};$$

after which the vanishing of the sixth pair gives us

$$3_1 3_{2\,5} 3_3 3_4 4_{6\,7\,8},$$

the pyramid on the base 1234.

But if we do not insist on the vanescence of gamic pairs, and on an autopolar result, we can reduce the 8-edron readily to the 7-edral pyramid, by the evanescence of $\mathbf{3}_7 4_2\mathbf{3}_8 6_4$, which makes the face $\mathbf{4}_{7\,8}$, and then the convanescence of $3_3\mathbf{4}_2 6_4\mathbf{4}_8$; but the first of these two steps gives a heteropolar result. Generally any x-edron that has at most a p-ace or a p-gon can be reduced by single vanescences to the $(p+1)$-edral pyramid.

In the second of the above three 8-edra, the first pair, though at first sight it appears a vanescible pair, is not so. For putting s_n and f_n for the summit and face signed n, we see in the eighth and second pairs, or the nodal pairs, s_1 in f_1, and s_3 in f_3, s_3 in f_2, and s_2 in f_3. If now $\mathbf{3}_1 4_4\mathbf{4}_2 4_5$ evanesces, we have $f_1=f_2$, whence, s_2 is in f_3 through s_3, and s_3 in f_1 through s_1, so that $3_1\mathbf{4}_4 4_2\mathbf{4}_5$ is no longer convanescible, f_1 and f_3 through s_1 and s_2 being covertical, and in fact collateral faces.

Hence the only vanescible pairs are the second and third. As they stand, the summits read thus with their signatures,

$$3_1 4_2 4_3 4_4 4_5 3_6 3_7 3_8.$$

The vanescence of the third pair gives us the reading

$$3_1 5_{27} 3_3 3_4 4_5 3_6 3_8\ ;$$

and now the fourth pair has become the only vanescible one, giving the reading

$$3_1 4_{27} 3_3 4_{48} 3_5 3_6,$$

which by the vanescence of the sixth pair becomes

$$3_1 3_{27} 3_3 3_{48} 4_{56},$$

the pyramid on the base 1234.

If we had begun by making the second pair vanish,

$$3_1 4_2 4_3 4_4 4_5 3_6 3_7 3_8$$

would have reduced to

$$3_1 3_2 4_{36} 4_4 4_5 3_7 3_8,$$

for the third summit losing both the edges of the nodal angle, 36 and 32, can unite with the sixth to form a 4-ace only. The sixth pair vanishing, gives us

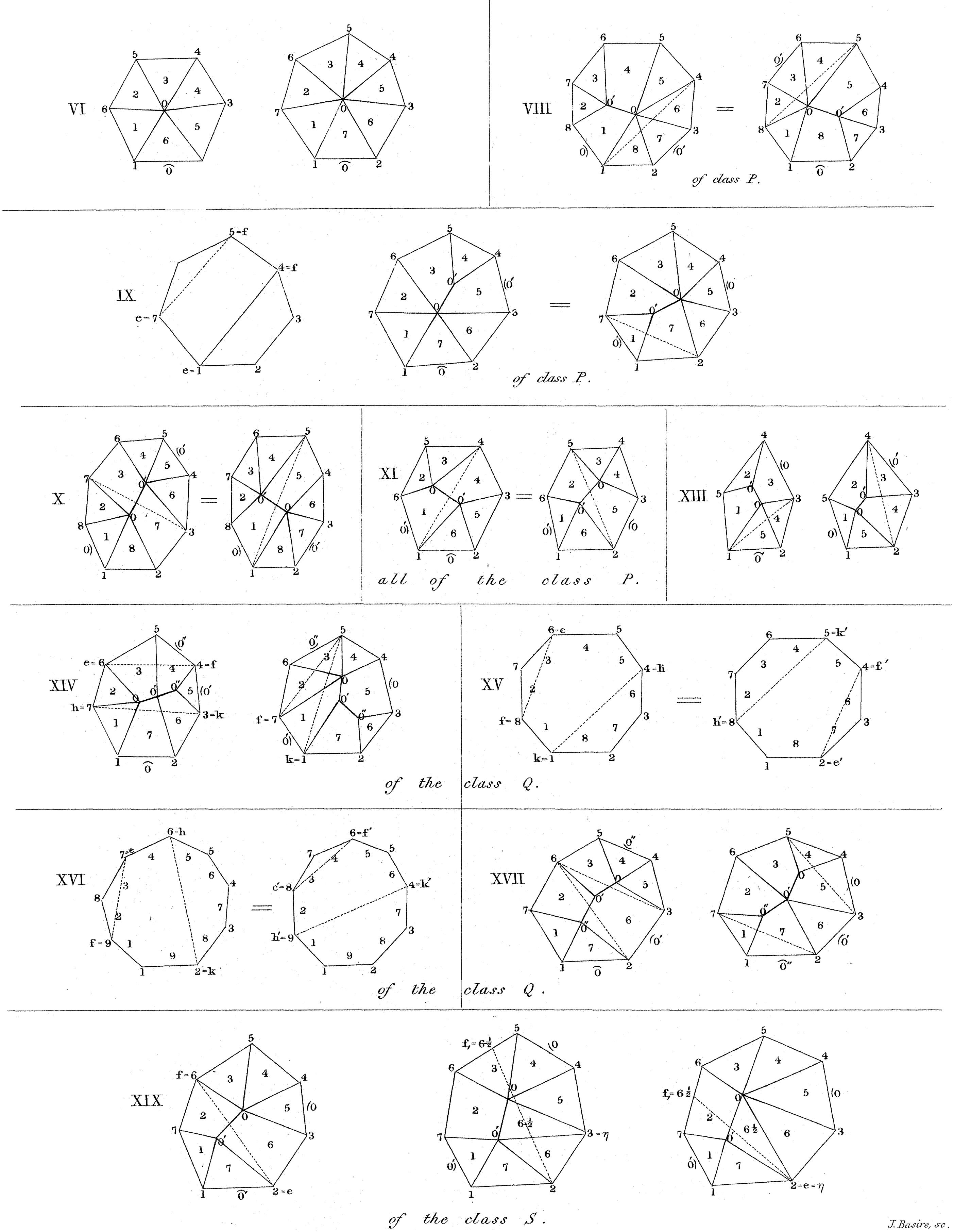
$$3_1 4_{28} 3_{36} 4_4 3_5 3_7,$$

which reduces, as before, to the pyramid on the base 1234 by the vanescence of the fifth pair.

The vanescible pairs of the third 8-edron are the second and the fifth, which give us two ways of reducing it to the pyramid on the base 1234. Or we can reduce it by the evanescence of $\mathbf{3}_5 4_4 \mathbf{3}_2 5_8$, $\mathbf{4}_4 4_1 \mathbf{3}_6 4_8$, and convanescence of $3_4 \mathbf{4}_1 3_6 \mathbf{5}_8$ and $3_3 \mathbf{4}_2 3_4 \mathbf{4}_8$ to the pyramid on pentagonal base.

XL. All the autopolars above given can be represented by square paradigms, showing all the faces, summits, edges and angles of the figure; for a closed polygon can be drawn through the summits of any of them. For example, the three last written have through their summits the circles 15238746, 15672384, 15238746; *i. e.* the duads of these circles occur among the non-contiguous duads of the subindices. If these circles be employed as directed in my paper "On the Representation of Polyedra," in the volume for 1856 of the Transactions of the Royal Society, the paradigms are easily written out. But I do not find that any mode of representation is so simple and distinguishing as that above given in art. XXXVI. and those following it.

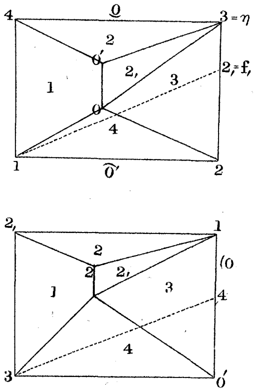
The figures in Plates XII., XIII., XIV., are intended to illustrate the Articles of this Memoir to which reference is made in them.

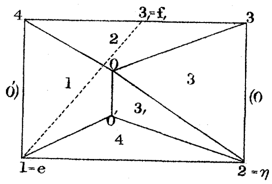

VI
VIII
of class P.
IX
of class P.
X
XI
XIII
all of the class P.
XIV
XV
of the class Q.
XVI
XVII
of the class Q.
XIX
of the class S.

J. Basire, sc.

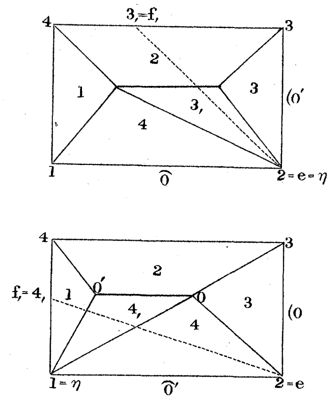
XX

&c.

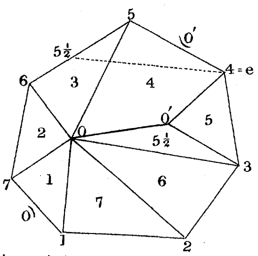




This is the S rejected in XXI.

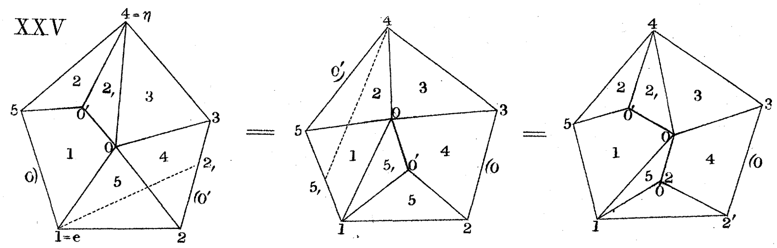


XXI



This is rejected from the class S in XXI *because of the vanescible pair,* 05 = 34 *and* 34 = 05.

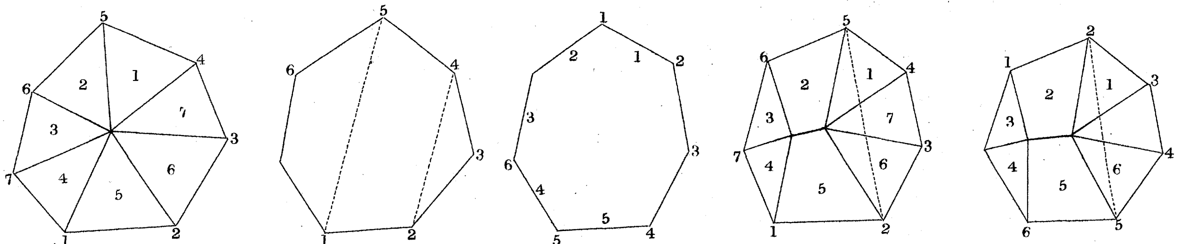
XXV



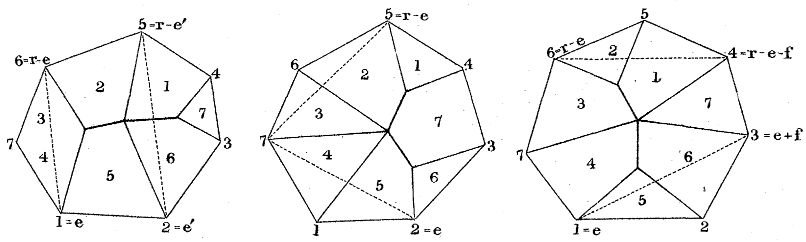
p. 203.

The second is a repetition of the first, and both are identical with the third.

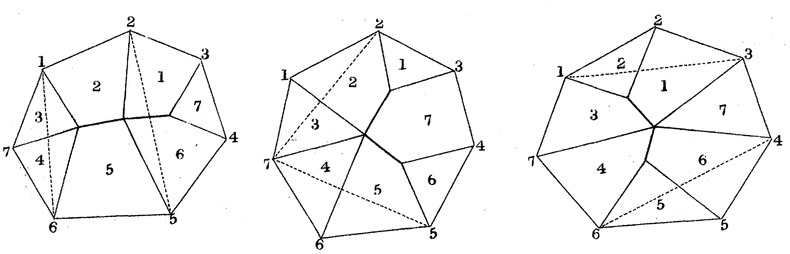
XXXI



enodal — *nodal* — *enodal* — *nodal*

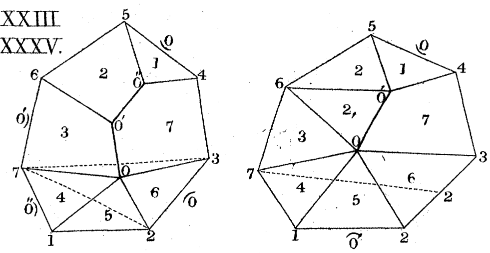


XXXII



all utrally autopolar.

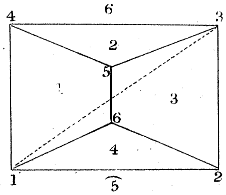
XXXIII & XXXV.

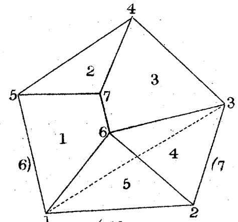


purely enodal.

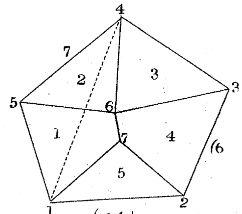
The autopolar 6-edron and 7 edra.

XXXVI

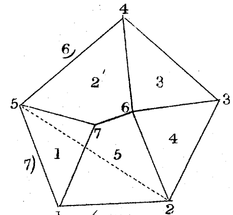




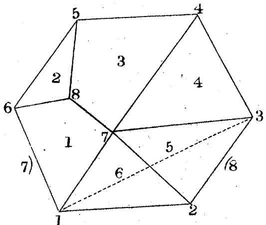
(13)



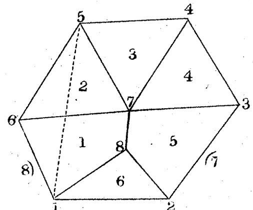
(14)



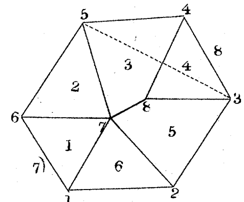
(25)



(13)



(15)



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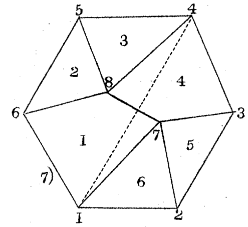
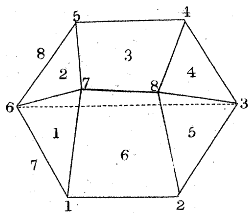
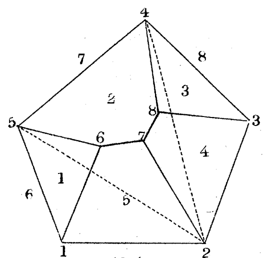


(14)

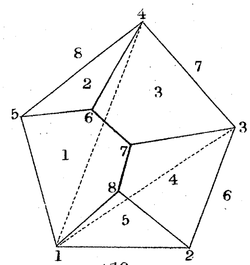
The 15 autopolar 8 edra; XXXVI &c.



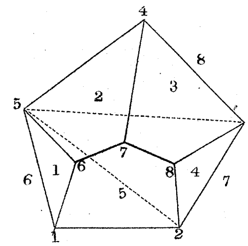
(36)



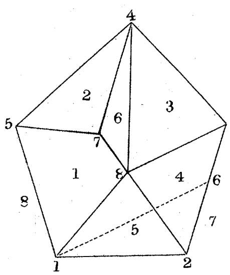
$\binom{24}{25}$



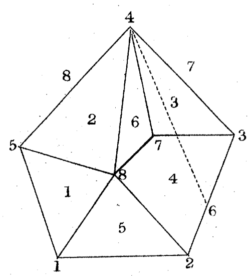
$\binom{13}{14}$



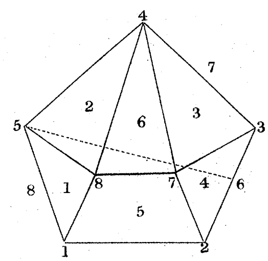
$\binom{52}{53}$



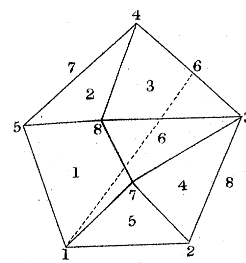
($12\frac{1}{2}$)



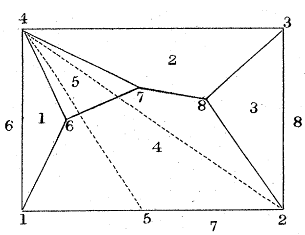
($42\frac{1}{2}$)



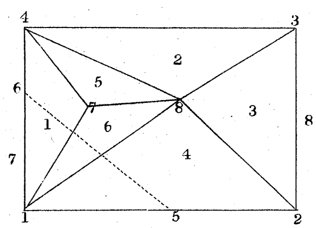
($52\frac{1}{2}$)



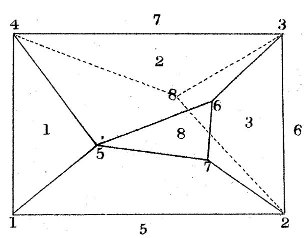
($13\frac{1}{2}$)



$\binom{42}{41\frac{1}{2}}$



$\binom{1\frac{1}{2}}{4\frac{1}{2}}$



(84, 82, 83.)

J. Basire, sc.